# Topic 1

## Real numbers and algebraic expressions
[sections 1.1 - 1.4]

Algebra: is a way to relate quantities without having to keep track of every single value

ex:

In order to encourage reading, your school offers to give you \$5 for each book you read

| Books | Money |
|---|---|
| 1 | 5 |
| 2 | 10 |
| 3 | 15 |
| etc | |

One nice way to summarize the following table is using variables.

For example, if you call

$B$ = # of books you have read

$M$ = money paid

Then the equation reads

$$M = 5B$$

However, someone could have written the equations as

$$D = 5B$$

where $D$ now represents the dollars you have been paid.

Hence in algebra, it is common to choose certain letters as the variables in order to use a more standard notation.

Typical letters for variables are x, y, z, w.

For example, the previous equation is more traditionally written as

$$y = 5x$$

The nice thing about doing this is that there are many other quantities that could satisfy this equation, like

$x =$ working weeks of the year

$y =$ work days during the year

The point of algebra is to study the mathematical features that these two examples share.

Variable = letter or symbol that represents a number and which can change for the problem under consideration

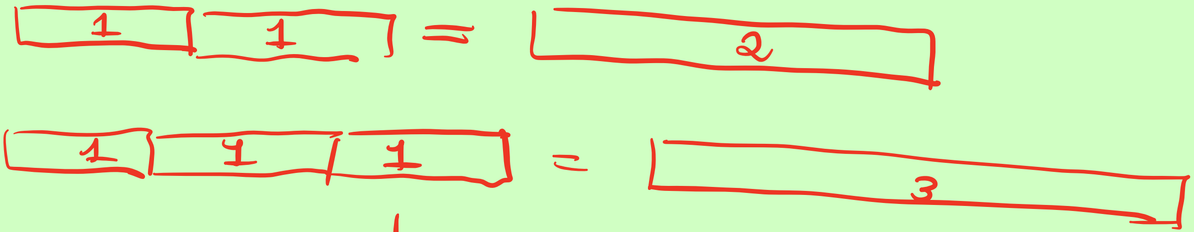
## Real numbers and the number line

Basic example:

suppose you have a ruler which you are using to measure length

Since it will be used to measure everything else, it can be assigned the number 1


the whole numbers correspond to the lengths which can be obtained by concatenating exact copies of the ruler




etc

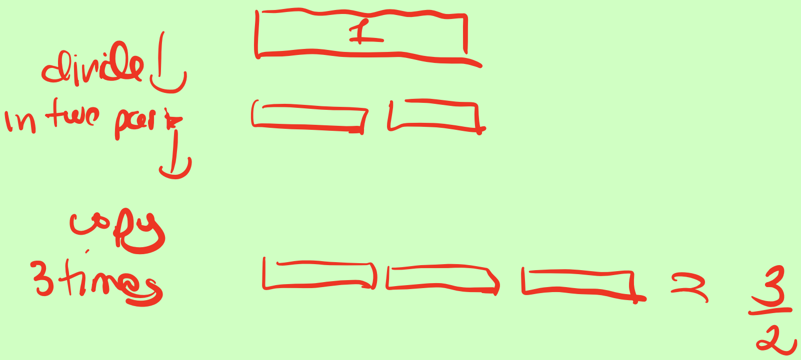
we use typically letters like a, b, m, n to refer to these numbers

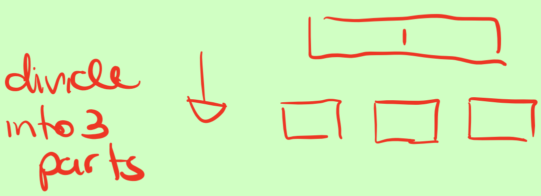
Rational numbers:

If you write

$\frac{a}{b}$ a → numerator, b → denominator

This means: divide the unit ruler into "b" pieces of equal length and make "a" copies of it



copy 2 time

$$\square\;\square = \frac{2}{3}$$

notice that

$$\frac{3}{2} \quad \text{and} \quad \frac{2}{3}$$

are related in that the numerator and denominator are switched.

These numbers are called reciprocals

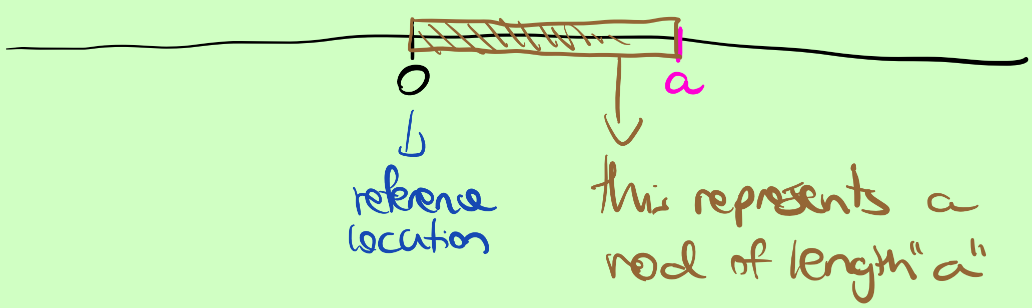
**Def:** the reciprocal of $\frac{a}{b}$ is the number $\frac{b}{a}$

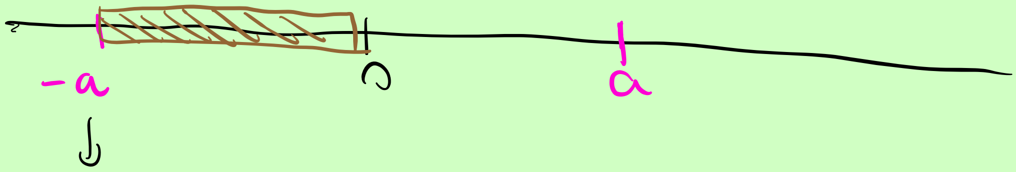
## Irrational numbers:

these correspond to the lengths which can't be written as a ratio.

for example $\pi$, $\sqrt{2}$, $e$ are irrational numbers (not obvious at all!!!)

## Number line






this represents the opposite of "a". it corresponds to a rod of length "a" but now placed to the left of "0"

Some important properties!

$$-0 = 0$$

$$--a = a$$

The integers are the numbers which correspond to the whole numbers and their opposites

so they include

$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

## The absolute value

$$|a|$$

is the length of the rod corresponding to "a"

examples

$$|3| = 3$$

$$|-5| = 5$$

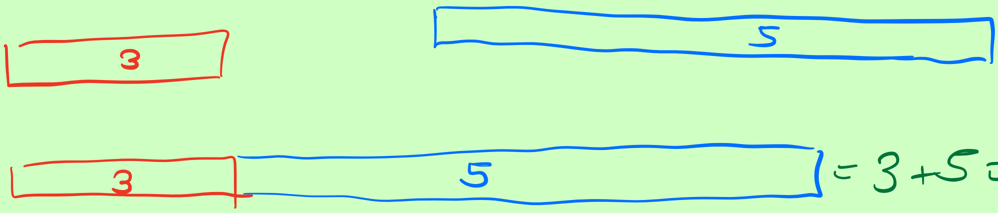
$$|0| = 0$$

in fact

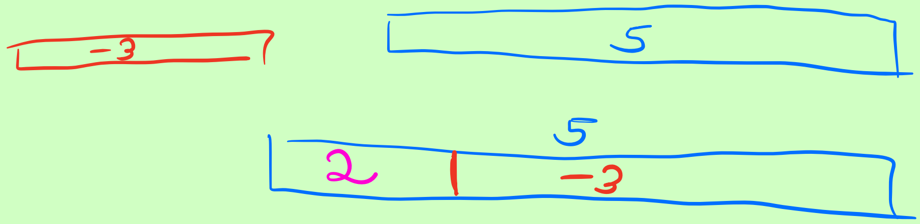
$$|a| = \begin{cases} a & \text{if } a \text{ is positive or } 0 \ (a \geq 0) \\ -a & \text{if } a \text{ is negative } (a < 0) \end{cases}$$

## Operations Between numbers;

Addition corresponds to concatenation




$= 3+5 = 8$




$-3+5=2$

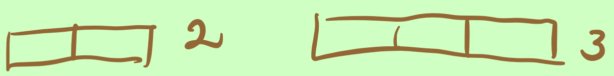
In general if $a, b$ are positive numbers then

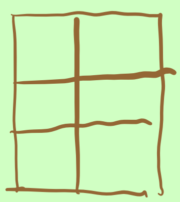
$$a-b$$

corresponds to $a+(-b)$ and corresponds to removing "b" units from "a"

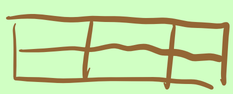
## Multiplication of numbers

If "a", "b" are positive then ab corresponds to the area of a rectangle with sides "a", "b" or alternatively to adding "a" with itself "b" times



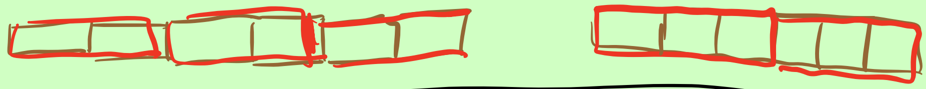



$2\cdot 3=6=3\cdot 2=$



or

$$2\cdot 3 = 2+2+2 = 3\cdot 2 = 3+3$$



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To multiply a positive and negative number, one can use this same rule adding the negative number as many times as determined by the positive number

$$(-2)\cdot(3) = -2 + -2 + -2 = -6$$

In fact,

$$\boxed{(-a)(b) = -(ab)}$$

---

To multiply two negative numbers, one uses the properties

$$--a = a, \quad -a = (a)(-1)$$

For example

$$(-2)(-3) = (2)(-1)(-1)(3) = (2)(--1)(3)$$
$$= (2)(1)(3)$$
$$= 6$$

Based on these examples, one becomes convinced that

positive times positive = positive

$+ \quad + \quad +$

positive times negative = negative

$+ \quad - \quad -$

negative times positive = negative

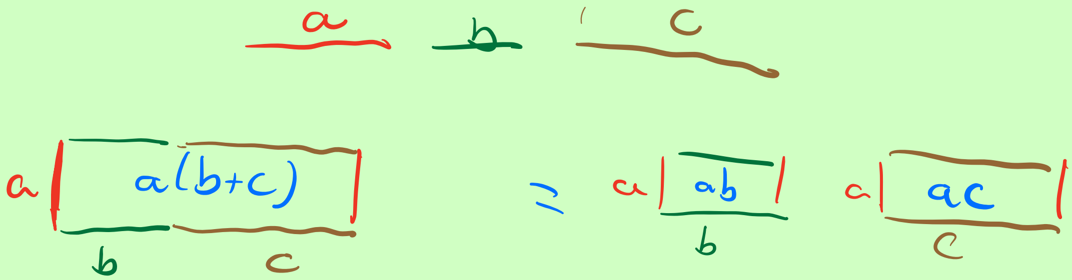
$- \quad + \quad -$

negative times negative = positive

$- \quad - \quad = +$

Multiplication and addition interact very nicely.
The most important property is the distributive property

$$a(b+c) = ab + ac$$




For example

$$3(x+1) = 3x + 3$$

This also works with negative numbers

$$3(x-1) = 3x - 3$$

It also works when more summands are involved

$$(3+x)(x-1)$$

$$= 3x - 3 + x^2 - x$$

$$= x^2 + 2x - 3$$

Project 2
Lines and Problem Solving
Part 1 (sections 1.5–1.8)

Linear equation: most basic type of equation for a variable "x".
For example:

$$\begin{cases} (a) & 5x+3=0 \\ (b) & -2+3x=2x+1 \\ (c) & 4x=3x \end{cases}$$

These equations are statements which become true or false after choosing specific values for x.

For example, in (a) you get by taking $x=0$ that

$$5(0)+3=0$$

or

$$0+3=0$$

or $$3=0$$

which is not true.

On the other hand, by taking $x=-\frac{3}{5}$ you get

$$5\left(\frac{-3}{5}\right)+3=0$$

or

$$-3+3=0$$

or $$0=0$$

which is true.

Hence, we say that $x = -\frac{3}{5}$ solves the equation $5x+3=0$.

In general for a linear equation, "x" always appears multiplied by some numbers, but never by other powers of x. Likewise,

| Linear | not linear |
|---|---|
| $2x+5=3$ | $x^2+3x+5=1$ |
| $-x+7=3x$ | $x^3=x$ |
| $4+2x=5$ | $\frac{x^2+1}{x}=3$ |

sometimes there are equations which do not seem linear but can be made linear after some work, like

$$\frac{2x+1}{3x-1}=5$$

we will see how to fix these later

Strategy for solving linear equations:

① Rewrite the equation so that the terms with "x" appear on the left side of the equations and the terms without it on the right side of the equation

ex:

$$-2+4x = 2x+1$$

$$-2+4x+2 = 2x+1+2$$

$$4x = 2x+3$$

$$4x-2x = 2x+3-2x$$

$$2x = 3$$

② Solve for "x" by making the number multiplying it equal to 1.

$$2x = 3$$

$$\frac{1}{2}\cdot 2x = \frac{1}{2}\cdot 3$$

$$\boxed{x = \frac{3}{2}}$$

---

ex 2:

$$3x = 2x$$

$$3x-2x = 2x-2x$$

$$\boxed{x = 0}$$

correct approach

---

$$3x = 2x$$

$$\frac{1}{2x}\cdot 3x = \frac{1}{2x}\cdot 2x$$

$$\frac{3}{2} = 1 \quad \text{(no solution!)}$$

incorrect approach! this fails because you can only multiply by $\frac{1}{2x}$ if you know that x can't be 0, which what we are trying to determine

---

$$\frac{2x+1}{3x-1} = 5$$

$$\frac{(2x+1)}{3x-1} \cdot (3x-1) = 5 \cdot (3x-1)$$

$$2x+1 = 15x-5$$

$$2x+1-1 = 15x-5-1$$

$$2x = 15x-6$$

$$2x-15x = 15x-6-15x$$

$$-13x = -6$$

$$\frac{1}{-13} \cdot (-13x) = \frac{1}{-13} \cdot (-6)$$

$$\boxed{x = \frac{6}{13}}$$

---

$$3x+1 = 3x+5$$

$$3x+1-3x = 3x+5-3x$$

$$1 = 5$$

$$1-1 = 5-1$$

$$0 = 4 \quad \text{(impossible!)}$$

---

Key fact: a linear equation can have
- one solution
- no solutions (previous example)
- be valid for every possible value of x (next two examples)

$$x = x$$

$$x - x = x - x$$

$$0 = 0 \quad \text{(always true)}$$

---

$$-2(x-1) = -4(1+x) + 2x + 6$$

$$-2x + 2 = -4 - 4x + 2x + 6$$

$$-2x + 2 - 2 = -4 - 4x + 2x + 6 - 2$$

$$-2x = -2x + 0$$

$$-2x = -2x$$

$$-2x + 2x = -2x + 2x$$

$$0 = 0$$

---

tricker example

$$ax = x$$

where a is a constant

$$ax - x = x - x$$

$$ax - x = 0$$

$$(a-1)x = 0$$

now if two numbers multiplied give 0, then there are two options

option 1

$$a - 1 = 0$$

$$\boxed{a = 1}$$

in this case the original equation becomes

$$x = x$$

option 2

$$\boxed{x = 0}$$

in this case the original equation becomes

$$a \cdot 0 = 0$$

which is always true | which is true regardless of the value of a

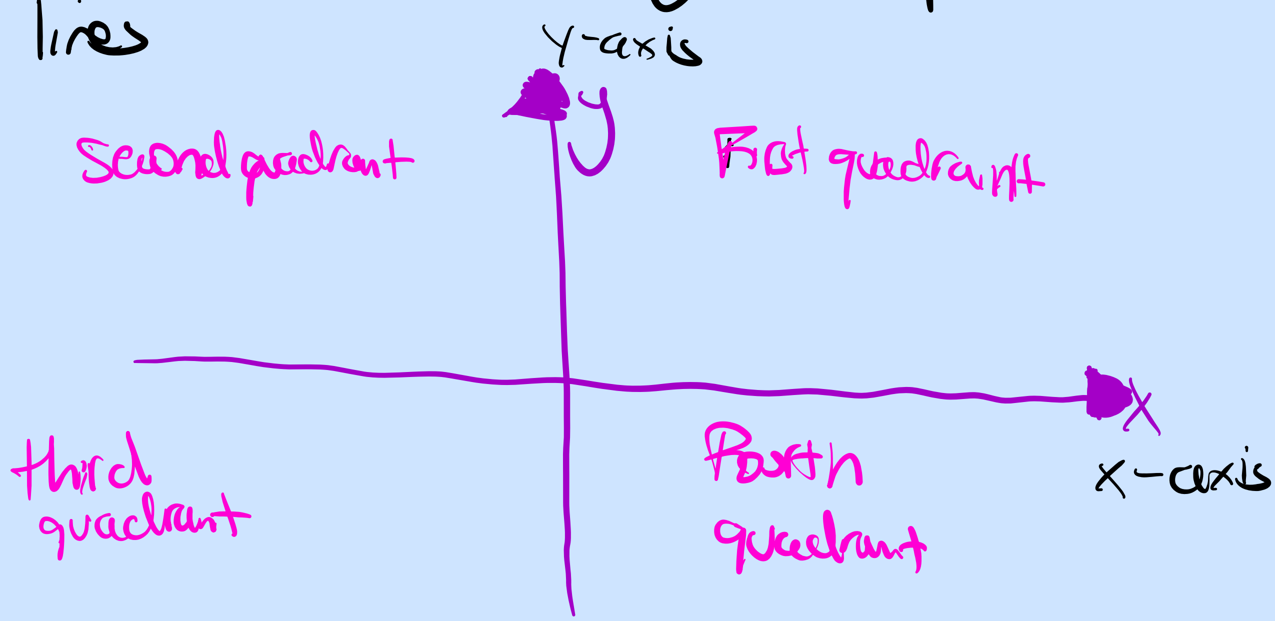
## Part 2

(sections 2.1 – 2.4)

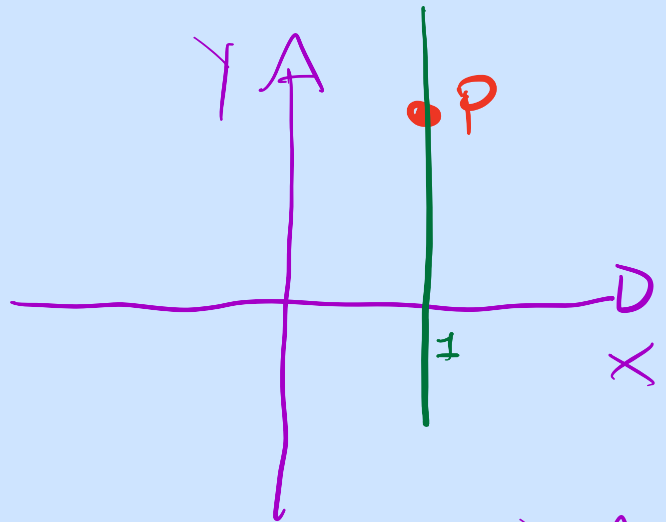
When two quantities are related via some sort of equation, like $y = x+1$, $y = x^2 - 1$, $x^2 + y^2 = 1$ etc, it is convenient to represent this relationship visually to draw conclusions more easily.

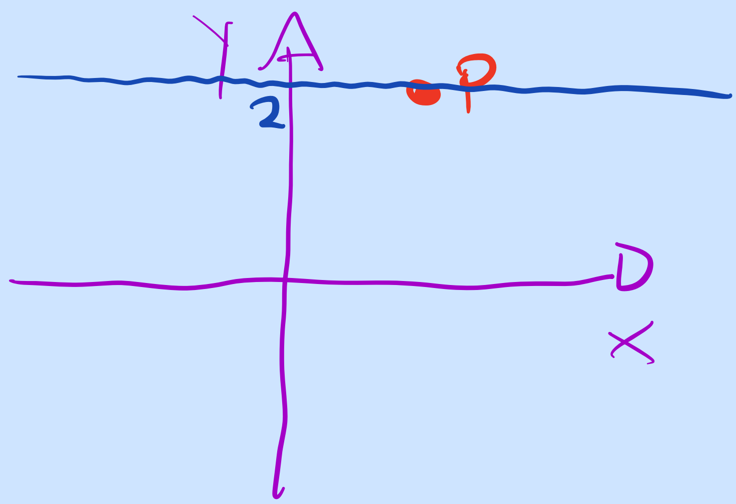
For this, we represent the information the cartesian coordinate system

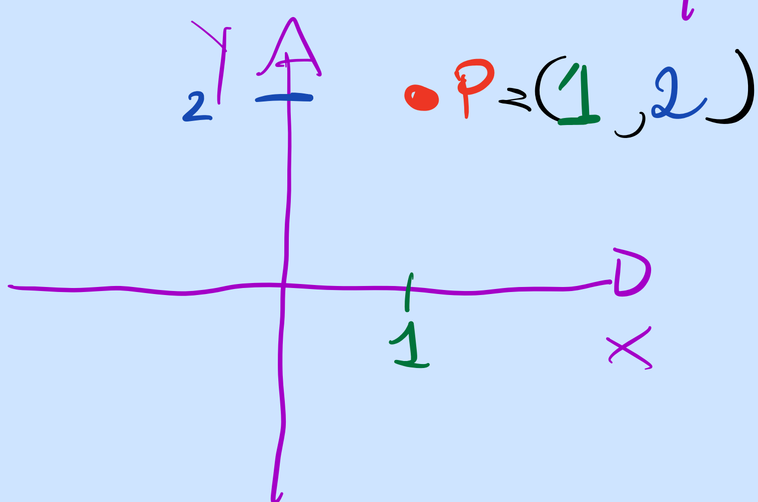
it consists of dividing the plane into 4 regions or quadrants, determined by two perpendicular number lines



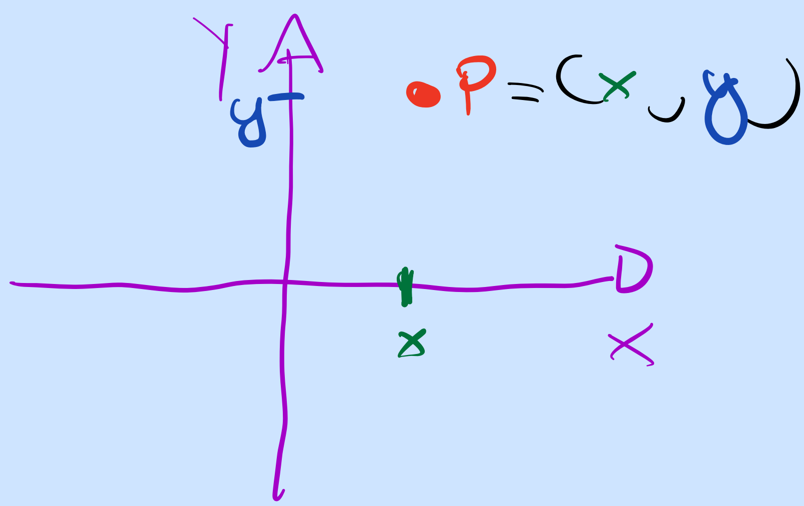
If you draw a point on the plane we assign two numbers to it by drawing lines parallel to each axis and passing through the point



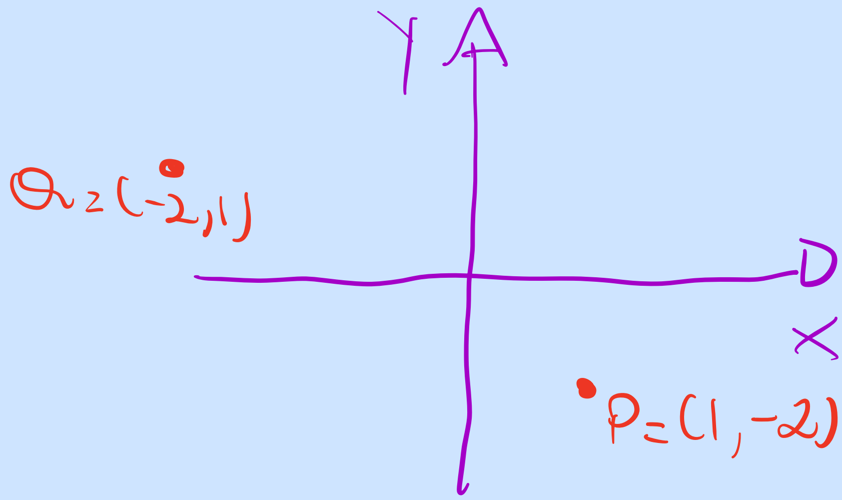



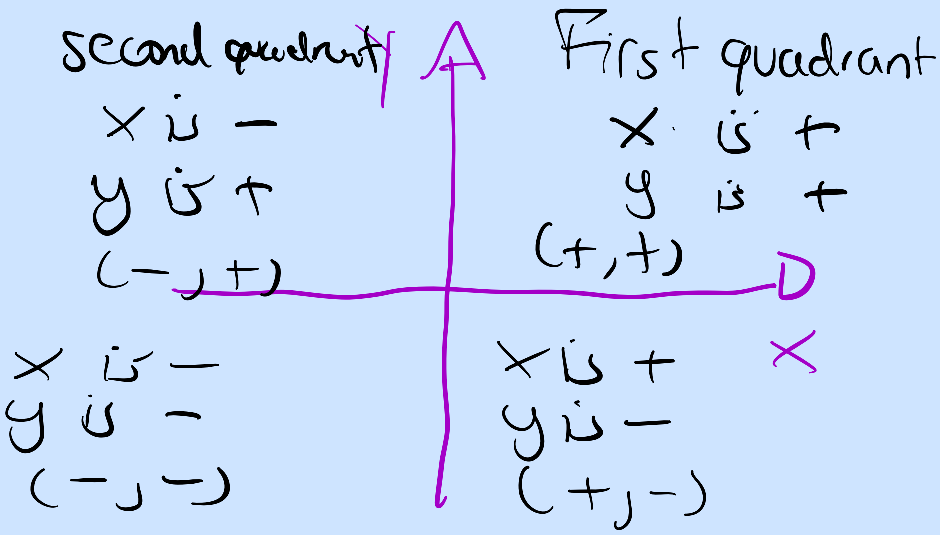

more generally we have



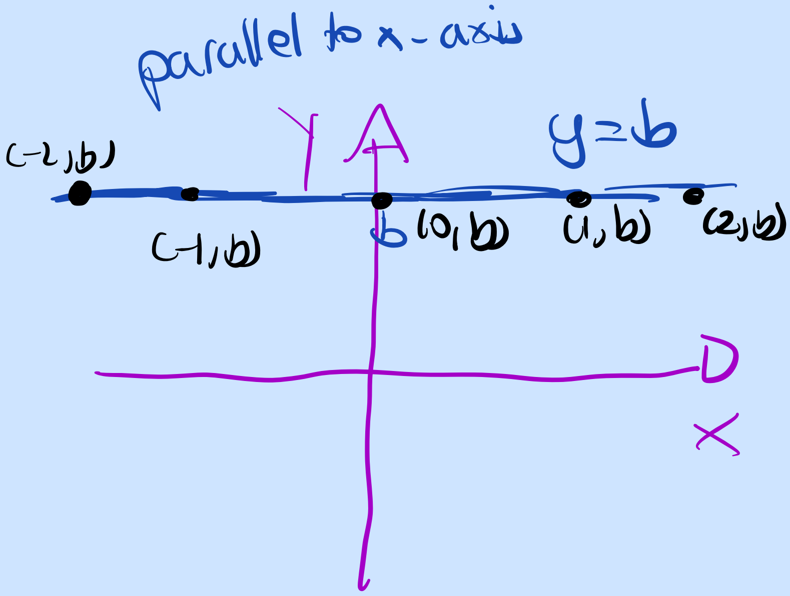
In general, the order is important

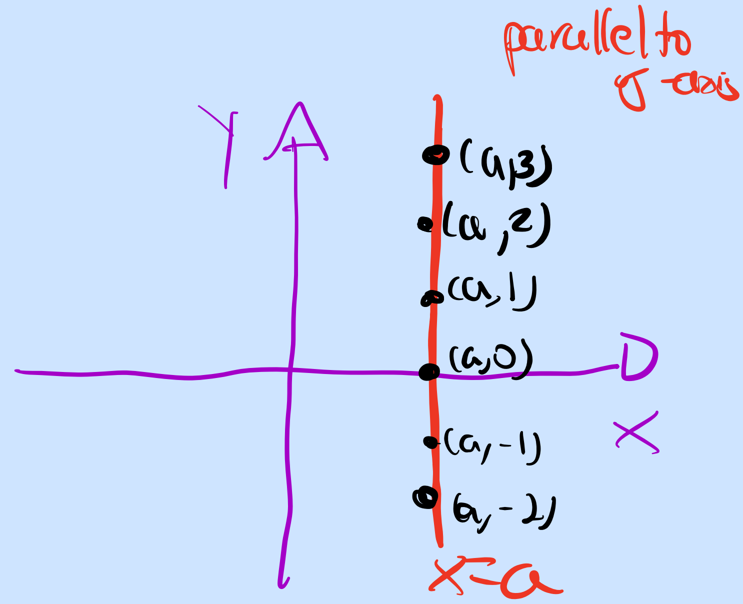



More generally we have



# Horizontal and vertical lines




## Linear equations and lines

a linear equation in two variables is an equation that can be written as

$$Ax + By = C$$

where $A, B, C$ are constants, with at least one of $A, B$ being non zero.

Examples

$2x + y = 3$

$-x + 2y = 5$

$x - y = 0$

$x = 5$

$y = -3$

$-2x = 4$

$-x - y = -5$

Special cases

$A = 0$

$Ax + By = C$ becomes $By = C$ or $\underbrace{y = \frac{C}{B}}_{\text{horizontal line}}$

---

$B = 0$

$Ax + \cancel{By} = C$ becomes $Ax = C$ or $\underbrace{x = \frac{C}{A}}_{\text{vertical line}}$

---

Since the special cases are easy to understand, we will now work assuming that $A, B$ are <u>both</u> non zero

in this case it is convenient to rewrite the equation so that $y$ is on the left side and $x$ on the right side, and it is solved for $y$:

$$-x + 2y = 5$$

$$-x + 2y + x = 5 + x$$

$$2y = 5 + x$$

$$\frac{1}{2} \cdot 2y = \frac{1}{2} \cdot (5 + x)$$

$$y = \frac{5}{2} + \frac{x}{2}$$

$$\boxed{y = \frac{x}{2} + \frac{5}{2}}$$

---

$$2x + y = 3$$

$$2x + y - 2x = 3 - 2x$$

$$y = 3 - 2x$$

$$\boxed{y = -2x + 3}$$

---

In general, we can always write the equation as

$$\boxed{y = mx + b}$$

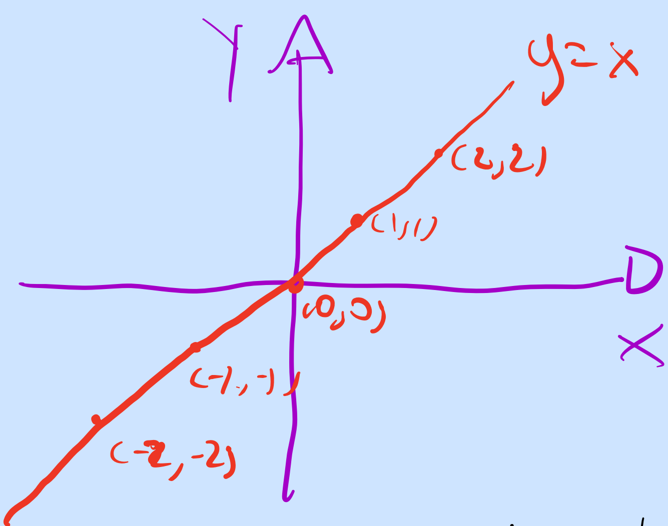
Here $m, b$ are two constants that store important information about the line.

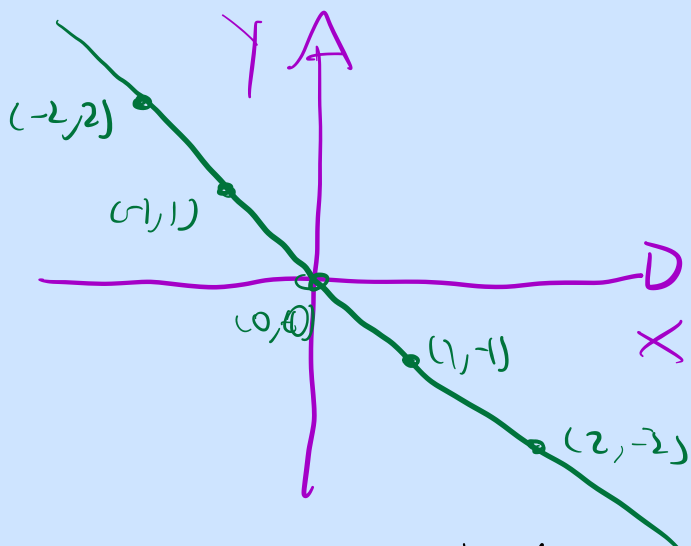
Let's start with a simpler situation where $b=0$. That is let's consider the equation

$$y = mx$$

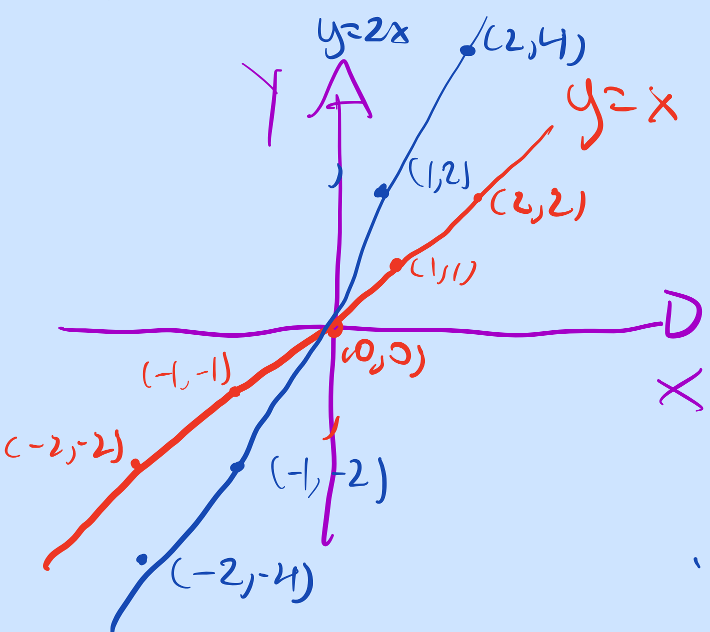
first, two important cases are when $m=1$, $m=-1$

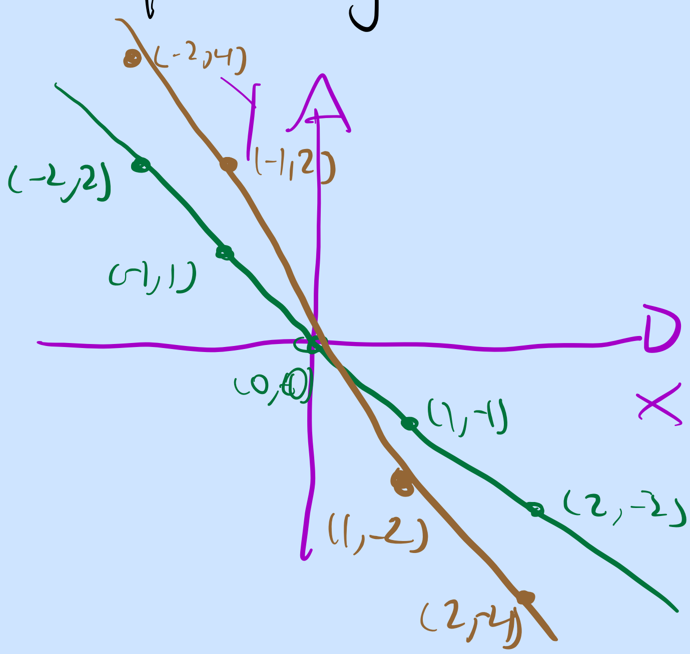
$y = x$ $\qquad\qquad$ $y = -x$




the others can then be compared against these

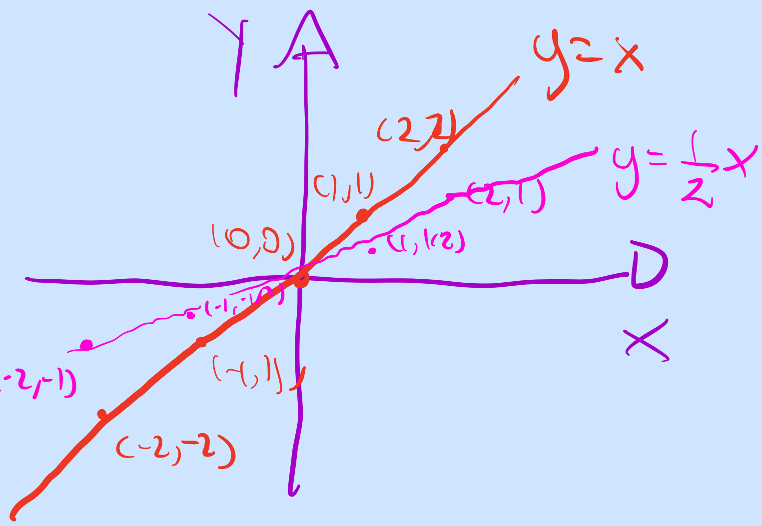


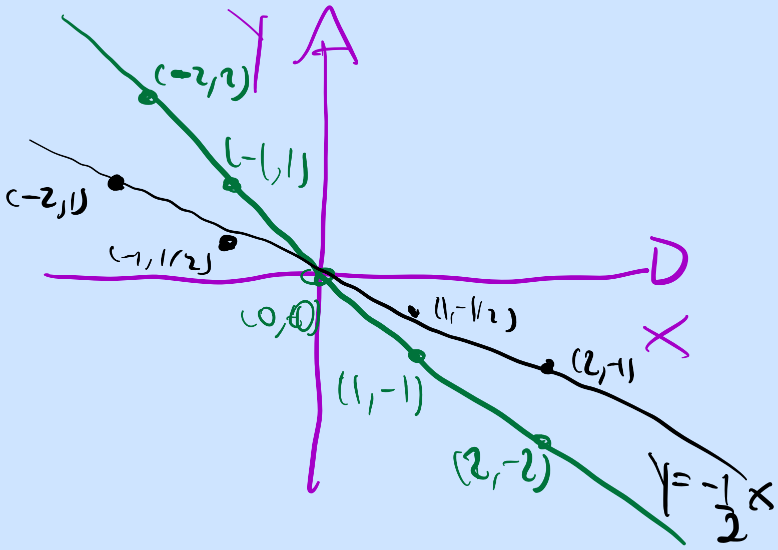



so $y=2x$ is closer to y axis than $y=x$

so $y=-2x$ is closer to y axis than $y=-x$

In general if $m>1$, then $y=mx$ or $y=-mx$ is closer to the y-axis than $y=x$ or $y=-x$



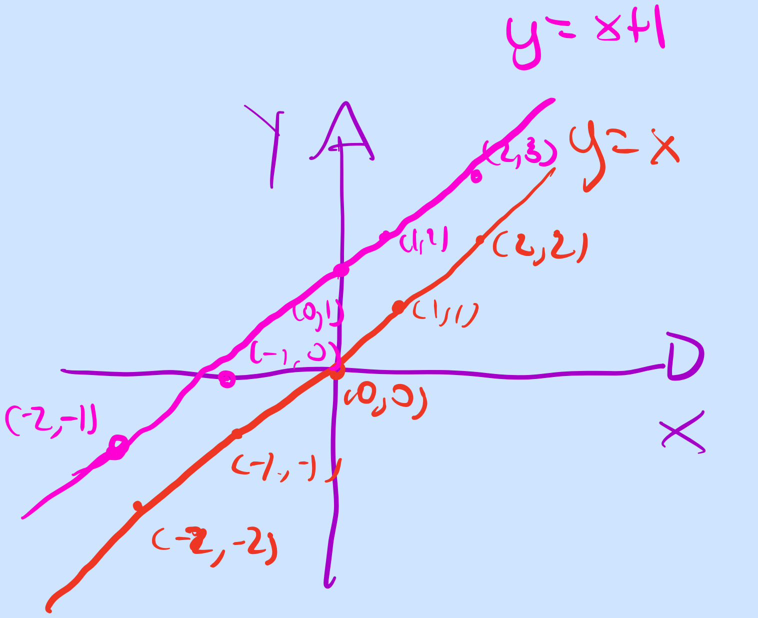

so $y=\frac{1}{2}x$ is closer to the x-axis than $y=x$

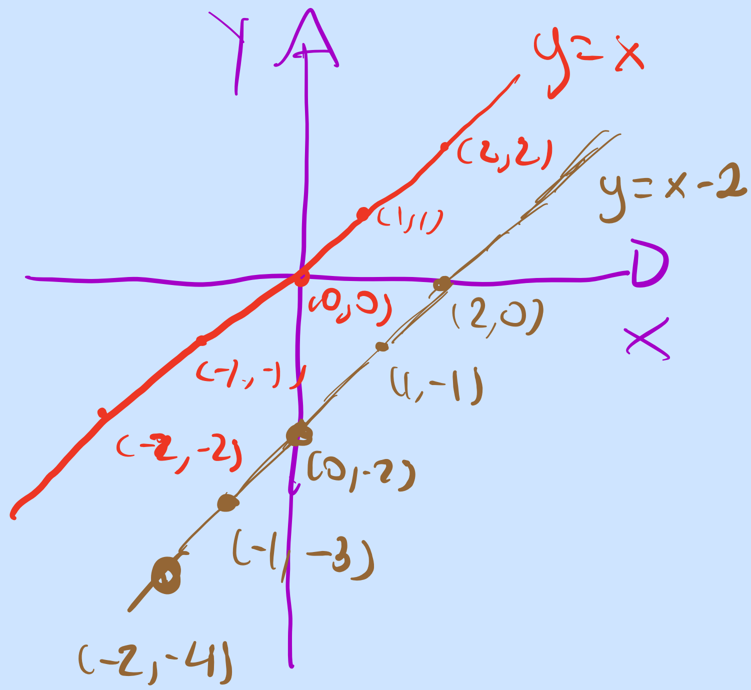
so $y=-\frac{1}{2}x$ is closer to the x-axis than $y=-\frac{1}{2}x$

In general if $0<m<1$ then $y=mx$ or $y=-mx$ is closer to the x-axis than $y=x$ or $y=-x$

General case $\boxed{y=mx+b}$

$y = x$



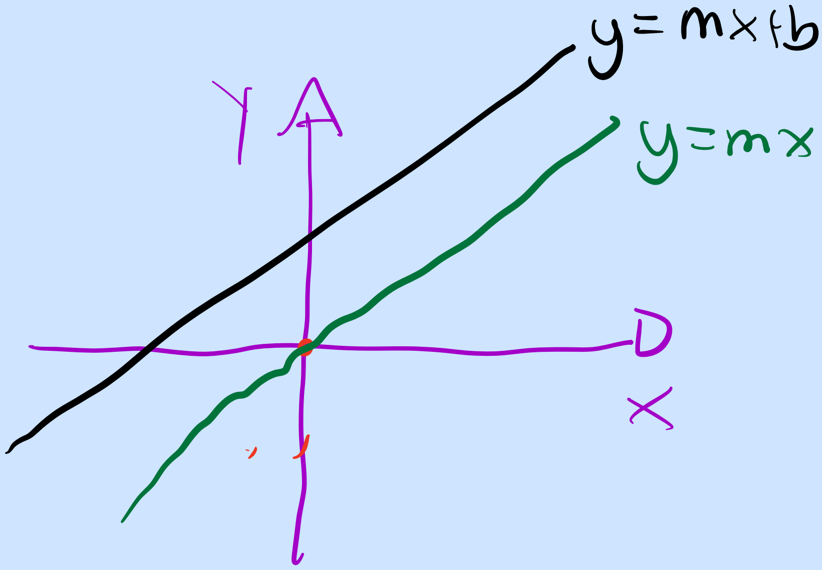

so we get a parallel line to $y=x$ but intersecting the y-axis at $y=1$

again another parallel line to $y=x$ but intersecting the y-axis at $y=-2$

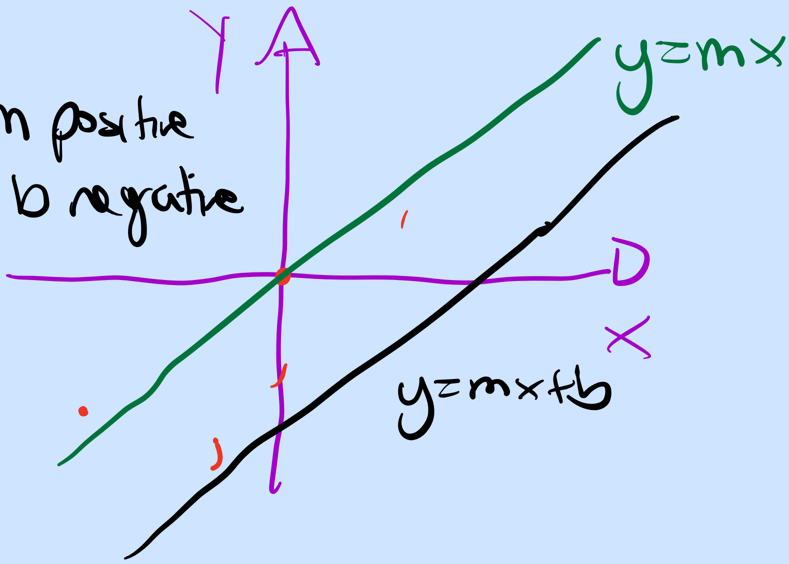
This is basically what happens with any line that is

$y=mx+b$ corresponds to a line parallel to $y=mx$ but intersecting the y-axis at $y=b$

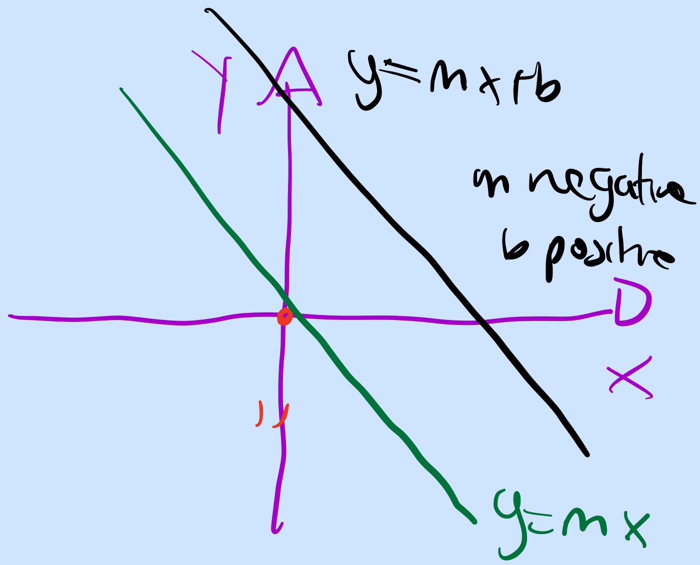
m, b positive




m positive
b negative

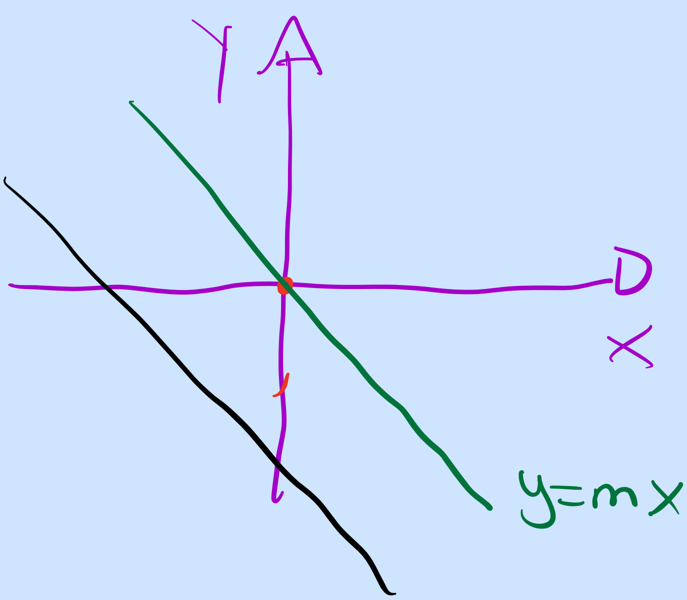



m negative
b positive




y=mx+b

m negative
b negative

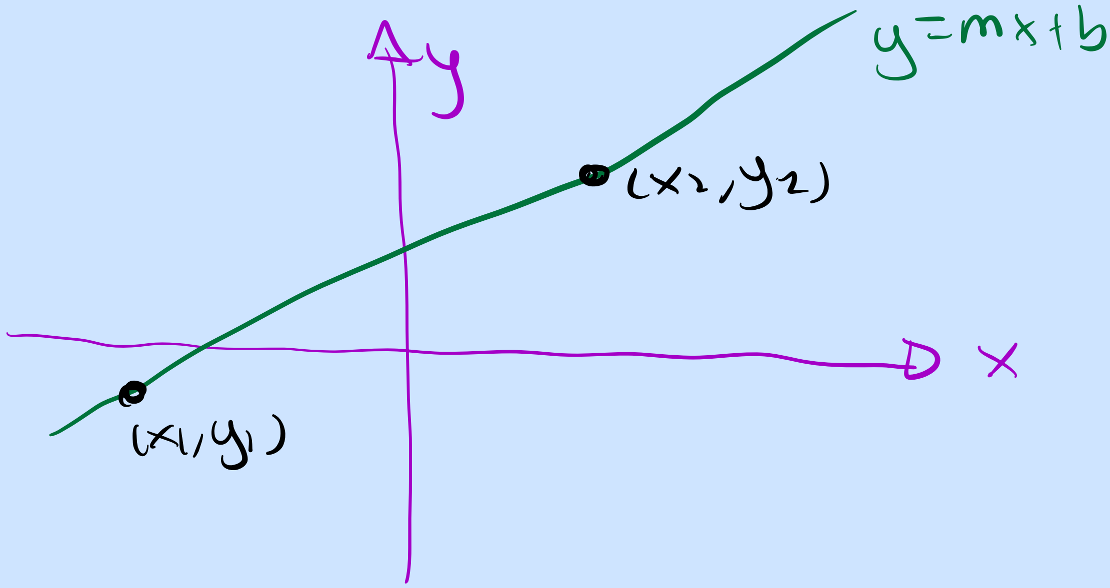


## How to find m?

Suppose you have a line with equation

$$y = mx + b$$

that passes through points $(x_1, y_1)$ and $(x_2, y_2)$




Since each point is on the line, each point satisfies the equation, that is

$$\begin{cases} y_1 = mx_1 + b & (1) \\ y_2 = mx_2 + b & (2) \end{cases}$$

We can use equation (1) to find "b"

$$\boxed{y_1 - mx_1 = b}$$

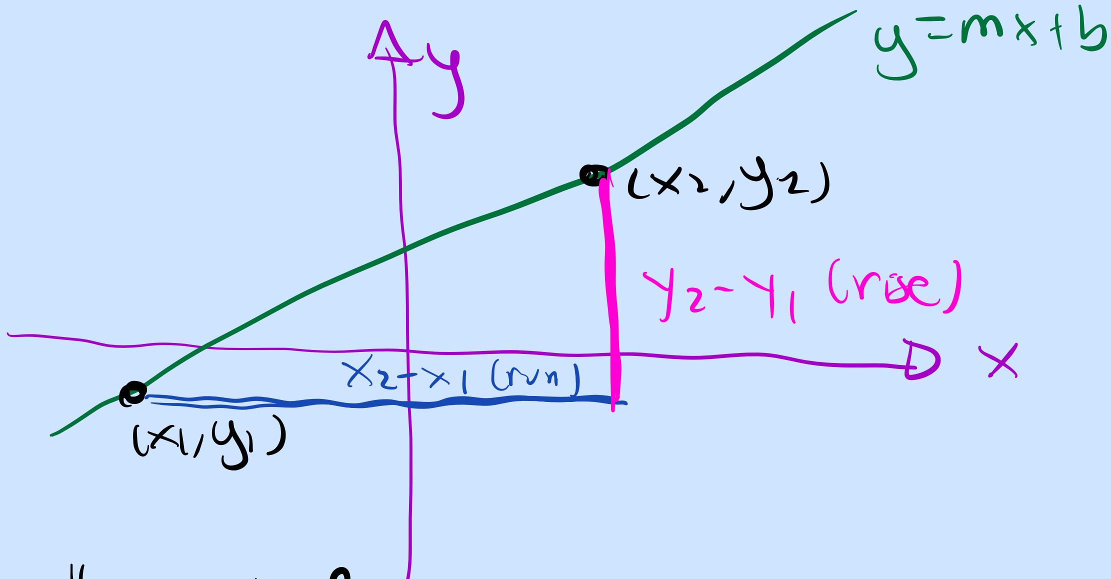
and replace on the second equation (2)

$$y_2 = mx_2 + y_1 - mx_1$$

$$y_2 - y_1 = mx_2 - mx_1$$

$$y_2 - y_1 = m(x_2 - x_1)$$

$$\boxed{\frac{y_2 - y_1}{x_2 - x_1} = m}$$




so "m" can be found as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

m is called the slope of the line

If "m" is positive the line is increasing

If "m" is negative the line is decreasing example!

Find the equation of the line passing through the points $(1,-1)$ and $(3,4)$

call

$(x_1,y_1) = (1,-1)$
$(x_2,y_2) = (3,4)$ } it doesn't matter which point is called $(x_1,y_1)$ and which one $(x_2,y_2)$

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{4-(-1)}{3-1} = \frac{5}{2}$$

Then the equation of the line must be

$$y = mx + b$$

or

$$y = \frac{5}{2}x + b \quad (*)$$

to find "b", substitute any of the points into the equation. For example, substituting $(1,-1)$ into the equation $(*)$ we find

$$-1 = \frac{5}{2}(1) + b$$

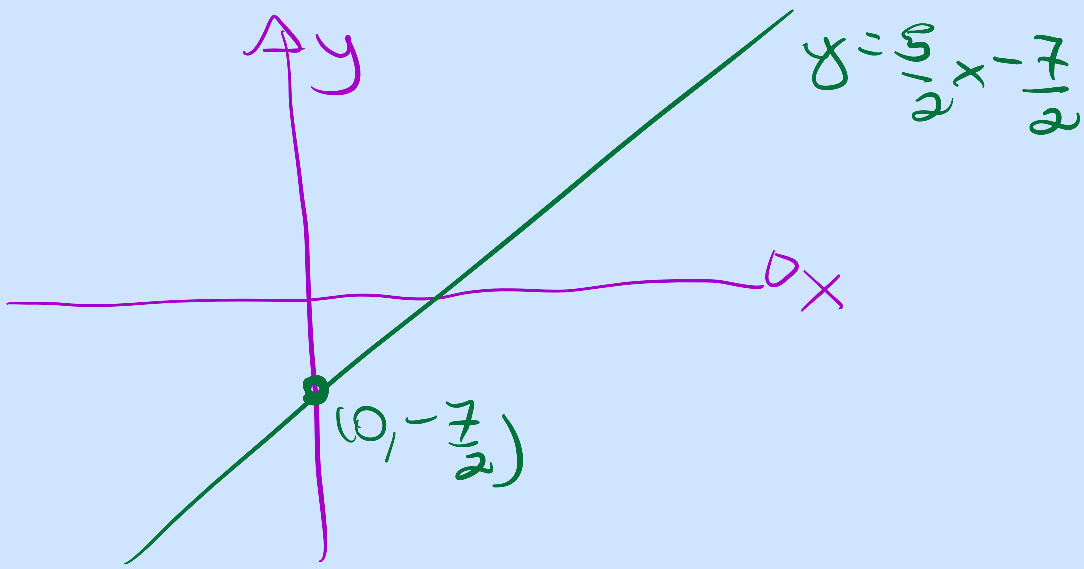
$$-1 = \frac{5}{2} + b$$

$$-1 - \frac{5}{2} = b$$

$$\boxed{-\frac{7}{2} = b}$$

Hence the equation of the line is

$$y = \frac{5}{2}x - \frac{7}{2}$$

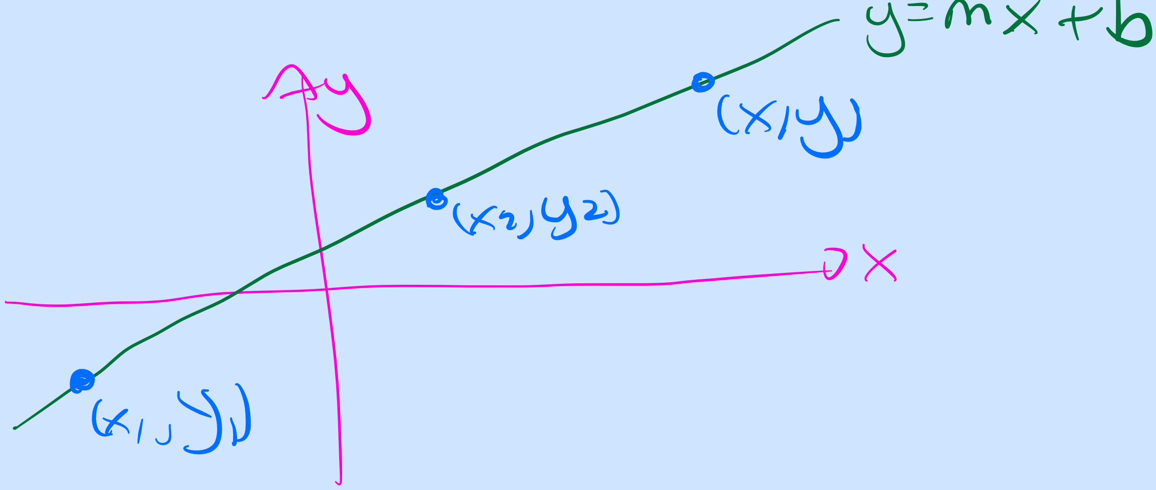



Remark: the equation

$$y = mx + b$$

is called the point intercept equation of the line

Another way to write the equation of the line is the so called point-slope form




Any two points can be used to find the slope so

$$\frac{y - y_1}{x - x_1} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Taking

$$\frac{y - y_1}{x - x_1} = m$$

we can write this equation as

$$y - y_1 = m(x - x_1)$$

# Project 3

$\boxed{\text{sections } 3.2,\ 3.6}$

As explained before, for a linear equation in two variables like

$$y = 2x+1 \quad \text{or} \quad 3x - y = 5$$

one can assign a value to one of the variables, and after doing this, the value of the other variable is determined by the equation.

However, one can also work with two linear equations at the same time. For example, we could try to solve

$$y = 2x+1 \quad \text{AND} \quad 3x - y = 5$$

This means that we are trying to find values for $x, y$ that satisfy both equations simultaneously. This is much harder to do.

For example, taking $x=0$ would give the equations

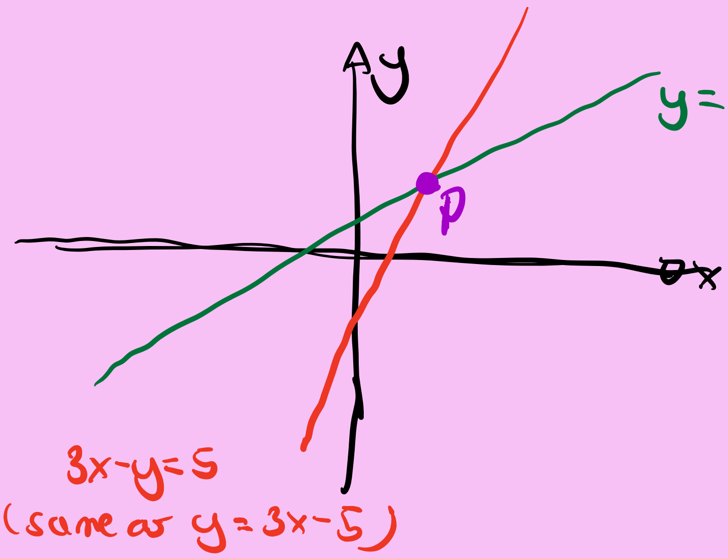
$$\left\{\begin{array}{l} y = 2x+1 \\ y = 2\cdot 0 + 1 \\ y = 0+1 \\ y = 1 \end{array}\right. \quad \text{AND} \quad \begin{array}{l} 3x - y = 5 \\ 3\cdot 0 - y = 5 \\ 0 - y = 5 \\ y = -5 \end{array}$$

Since we get different values for "y", this means that choosing $x=0$ can't give any y-values that will make both equations be satisfied.

In general

When solving a system of two linear equations trial and error is a BAD strategy.

To see what is going on, we need pictures




The lines are not parallel since their slopes are different, and therefore there will be a point of intersection P.

Therefore, finding a solution to the equations

$y=2x+1$ AND $3x-y=5$

corresponds to finding the point of intersection of the lines.

Since the x, y values must be the same regardless of which line equation we use, we can equate the y values

$y=2x+1$ and $y=3x-5$

Hence

$$2x+1=3x-5$$

$$2x+1-1=3x-5-1$$

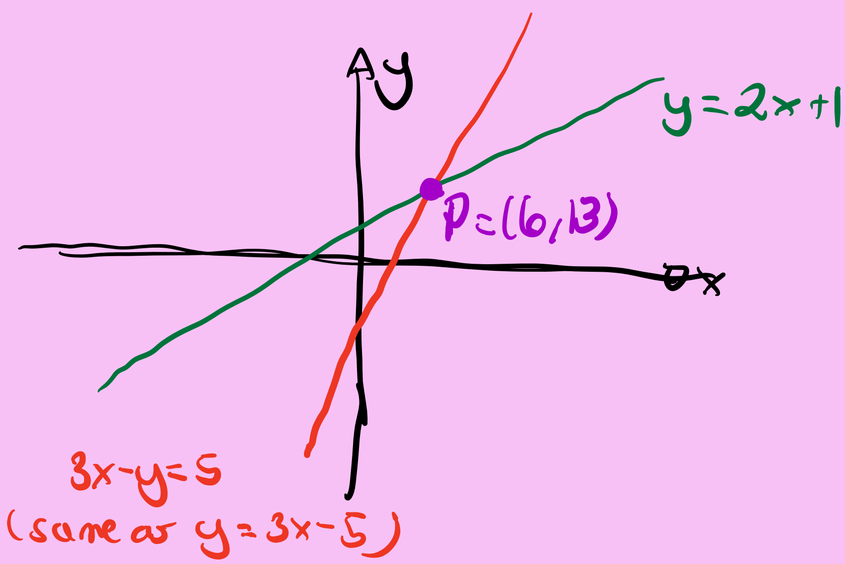
$$2x = 3x - 6$$
$$2x - 3x = 3x - 6 - 3x$$
$$-x = -6$$
$$\boxed{x = 6}$$

Once we find $x=6$, we can substitute back in any of the equations to find $y$

$$y = 2x+1$$
$$y = 2\cdot 6 + 1$$
$$y = 12+1$$
$$\boxed{y = 13}$$

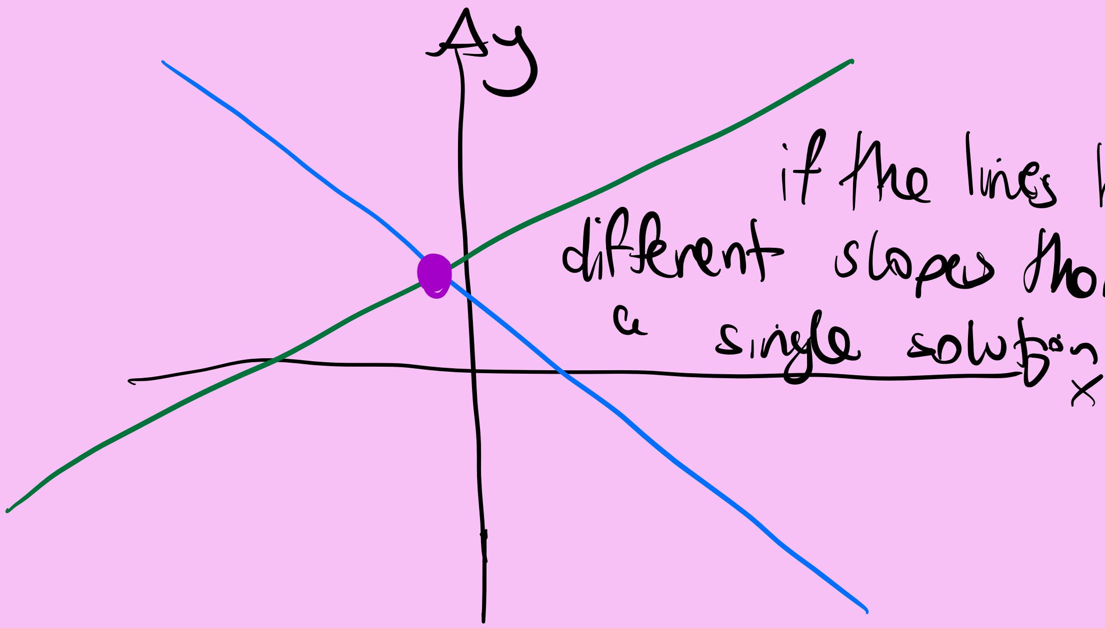
$$y = 3x - 5$$
$$y = 3\cdot 6 - 5$$
$$y = 18 - 5$$
$$\boxed{y = 13}$$

so the point of intersection is $(6, 13)$



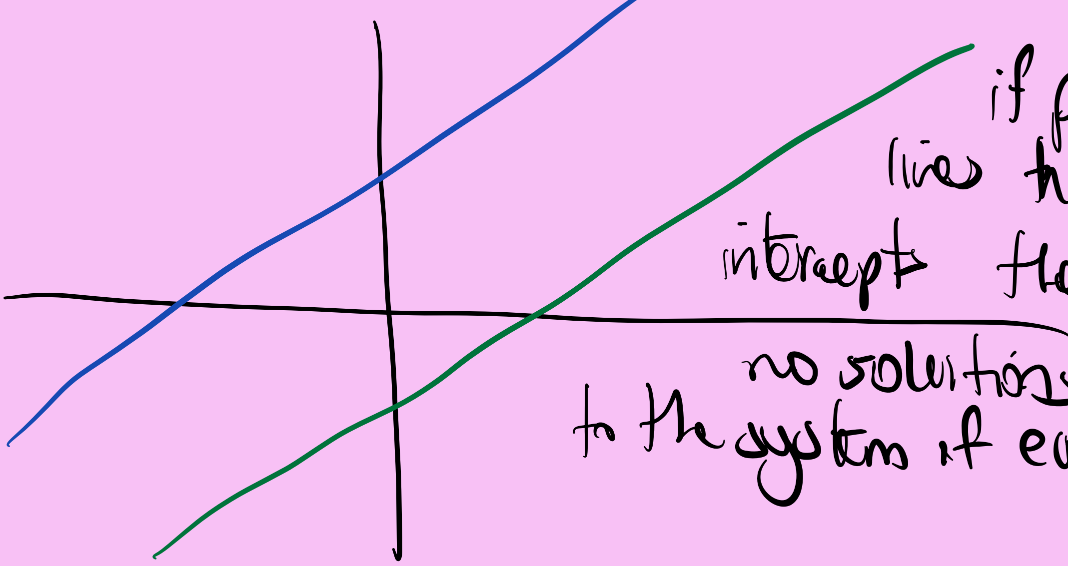
Solutions about systems of two equations in two variables

If two linear equations represent non parallel lines, then there is a unique solution to the system of equations




if the lines have different slopes there is a single solution

If two linear equations represent parallel lines then there either there are 0 solutions or infinitely many solutions

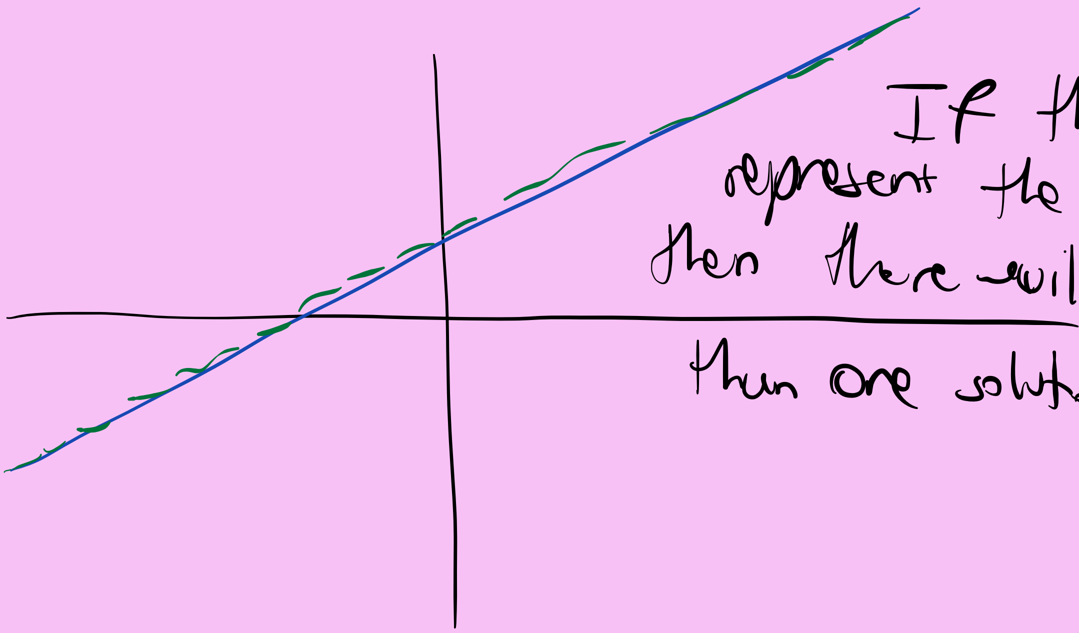


if parallel lines have different intercepts there are no solutions to the system of equations

example

$$\begin{cases} y = 3x+1 \\ y = 3x-2 \end{cases}$$

these equations can't be solved simultanously since it would give $1 = -2$.



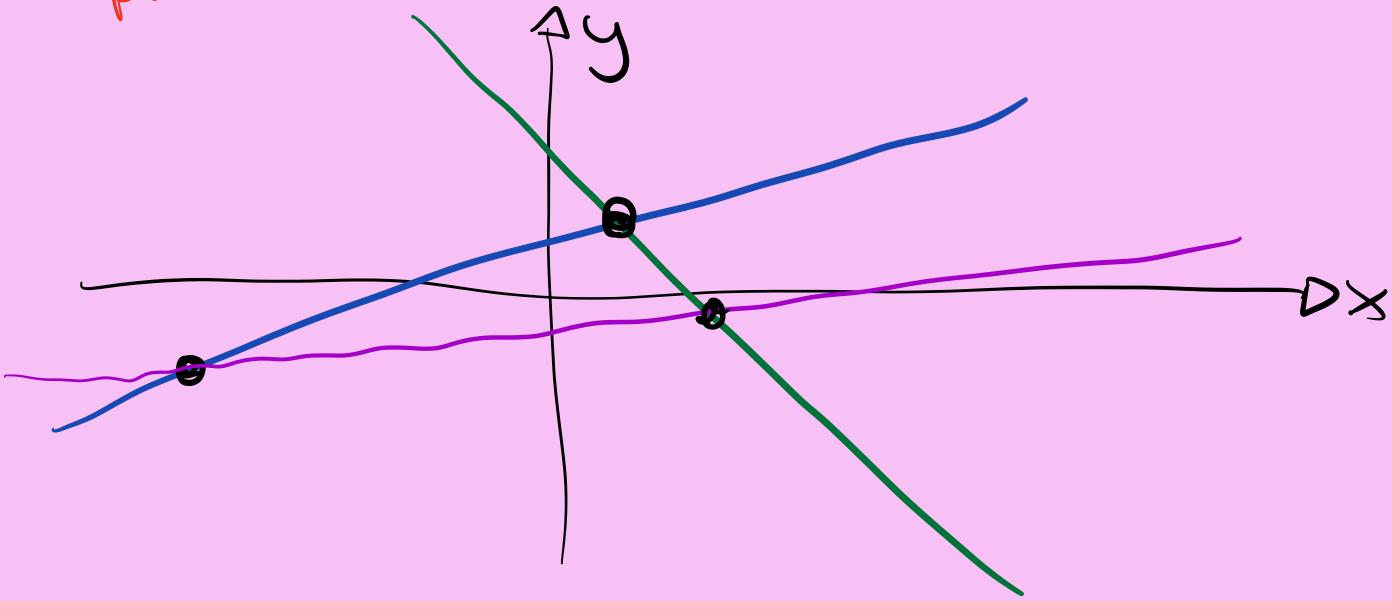
If the two equations represent the same line then there will be more than one solution

$$\begin{cases} y = 3x+1 \\ -6x + 2y = 2 \end{cases}$$

secretly represent the same line so there is more than one solution

# What about 3 equations?

option 1




If the lines intersect at distinct points there is no solution

example

$$\begin{cases} y = x + 1 \\ y = -x + 2 \\ y = -\frac{1}{2}x - \frac{1}{2} \end{cases}$$

The first two intersect when

$$x + 1 = -x + 2$$

$$2x = 1$$

$$x = \frac{1}{2}$$

thus at the point

$$\left(\frac{1}{2}, \frac{3}{2}\right)$$

The last two intersect when

$$-x+2 = -\frac{1}{2}x - \frac{1}{2}$$

$$-\frac{1}{2}x = -\frac{5}{2}$$

$$x = 5$$

Thus at the point

$$(5, -3)$$

The first and last lines intersect when

$$x+1 = -\frac{1}{2}x - \frac{1}{2}$$

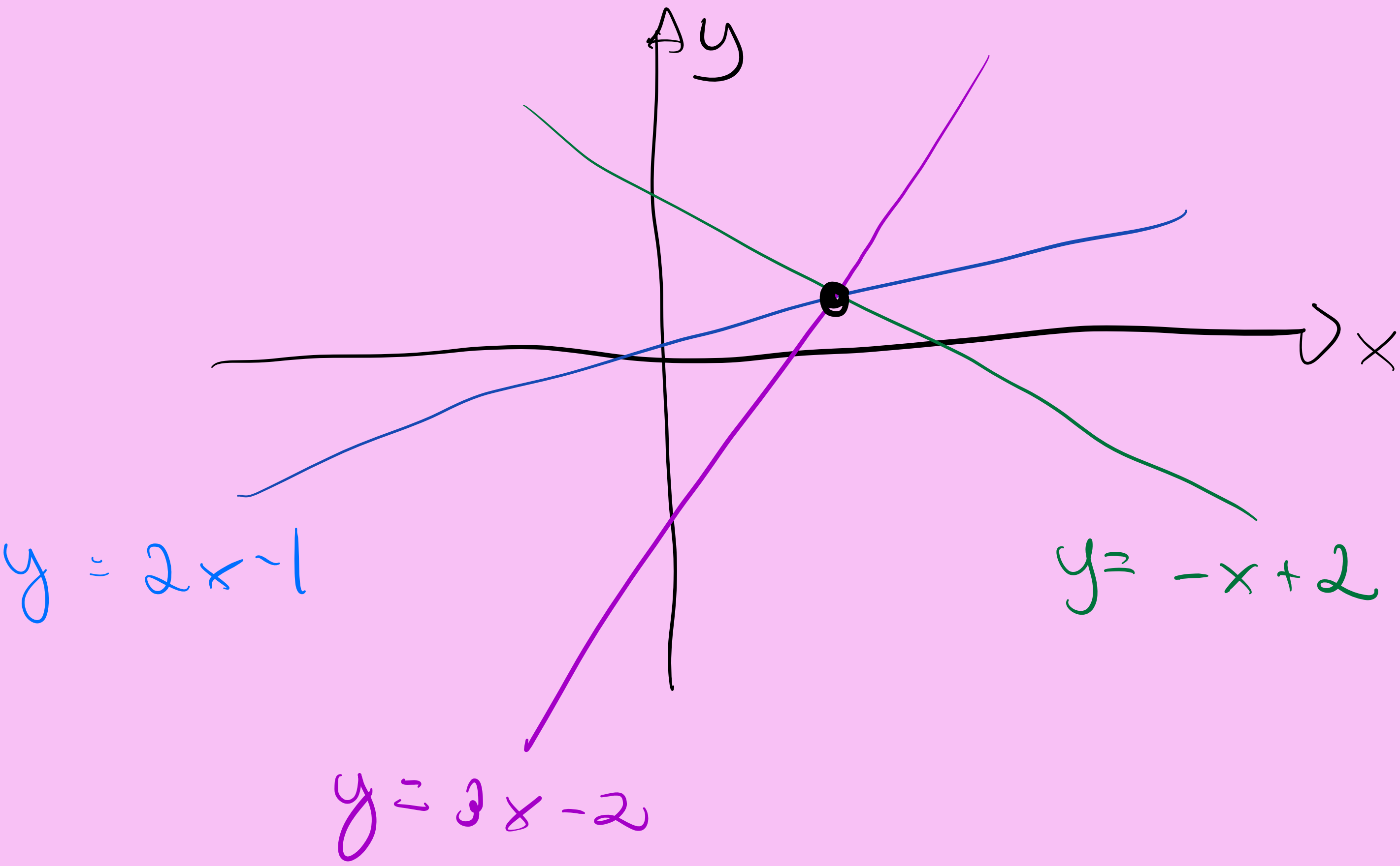
$$\frac{3}{2}x = -\frac{3}{2}$$

$$x = -1$$

Giving the point

$$(-1, 0)$$

option 2




$$\begin{cases} y=2x-1 \\ y=3x-2 \\ y=-x+2 \end{cases}$$

equating first two equations

$2x-1=3x-2$

$-x=-1$

$x=1$

so we get the point

$(1,1)$

equating last two equations

$3x-2=-x+2$

$4x=4$

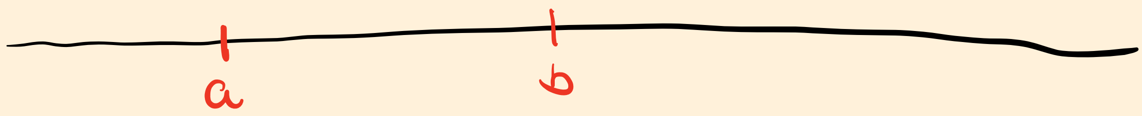
$x=1$

so we get the point

$(1,1)$

Project 4 (sections 4.1, 4.2, 4.3, 4.4)

Linear inequalities




$a \leq b$ means "a is less or equal to b"
so "a" is to the left of "b" or could equal "b"
one can also read this as "b is greater or equal to a" so "b" is to the right of "a" or could be "a"

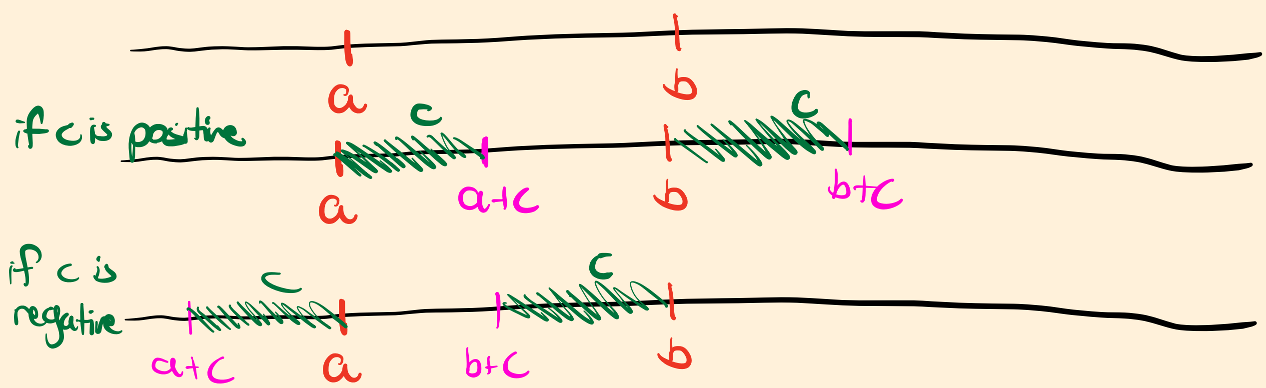
$a < b$ means "a is less than b"
so "a" is to the left of "b"
one can also read this as "b is greater than a"
so "b" is to the right of "a" or could be "a"

For example, we have
$3 < 5$, $-1 < 3$, etc

shifting property of inequalities

if $a < b$ then

$a + c \leq b + c$

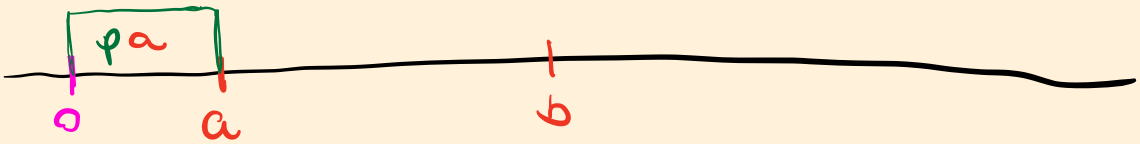


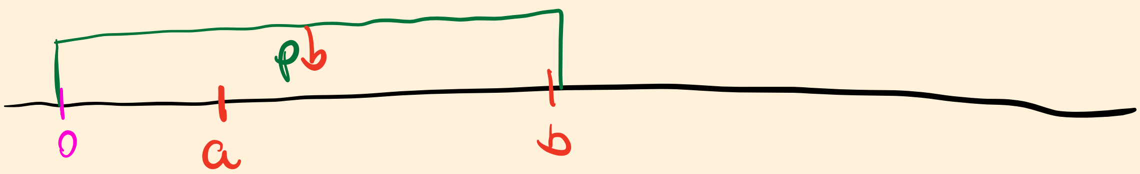
## multiplication property of inequalities

if $a \le b$ and $p$ is positive then

$$pa \le pb$$

will do a "picture proof" when both $a, b$ are positive




These two properties are enough to solve linear inequalities

## example 1

solve

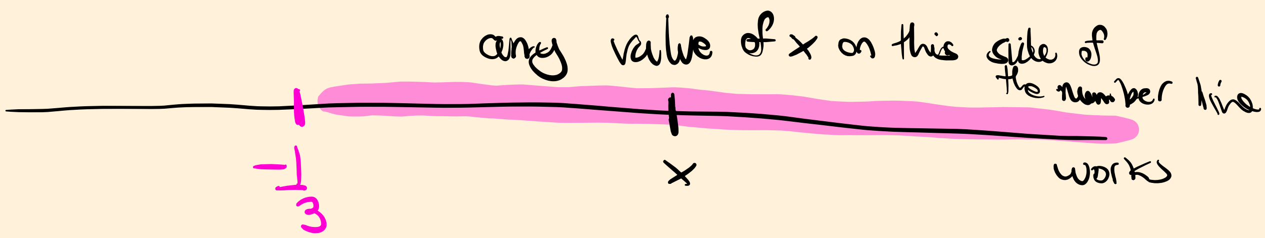
$$0 < 3x+1$$

$$0 - 1 < 3x + 1 - 1$$

$$-1 < 3x$$

$$\frac{1}{3}(-1) < \frac{1}{3} 3x$$

$$\boxed{-\frac{1}{3} < x}$$



Preferred method

equality method and test method

To solve $0 < 3x+1$

we solve instead the equality

$$0 = 3x+1$$

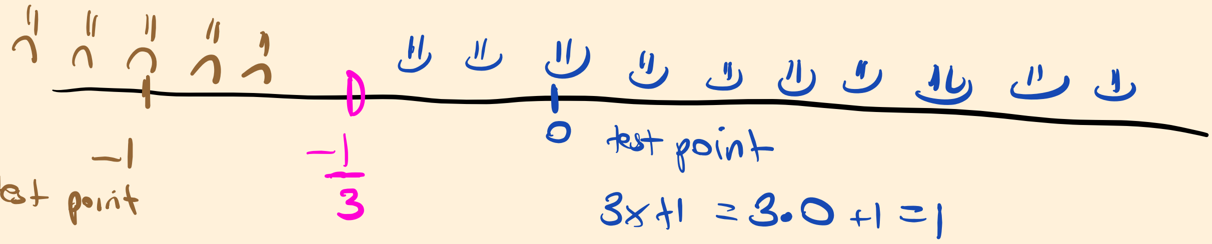
$$0-1 = 3x+1-1$$

$$-1 = 3x$$

$$\frac{1}{3}(-1) = \frac{1}{3}(3x)$$

$$\boxed{-\frac{1}{3} = x}$$

$$0 < 3x+1$$




test point

$3x+1$

$= 3(-1)+1$

$= -3+1$

$= -2$

does not satisfy inequality

test point

$3x+1 = 3 \cdot 0 + 1 = 1$

satisfies inequality

example 2

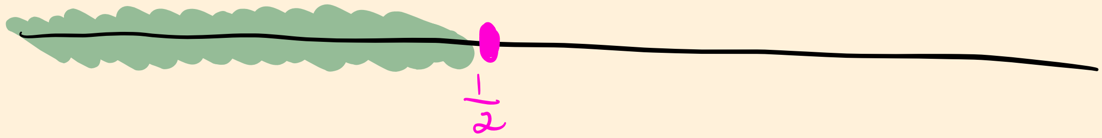
$$0 < -2x+1$$

$$0 + 2x < -2x+1+2x$$

$$2x < 1$$

$$\frac{1}{2} \cdot 2x < \frac{1}{2} \cdot 1$$

$$x \leq \frac{1}{2}$$




## Preferred method

to solve

$$0 < -2x+1$$

we solve the equality instead

$$0 = -2x+1$$

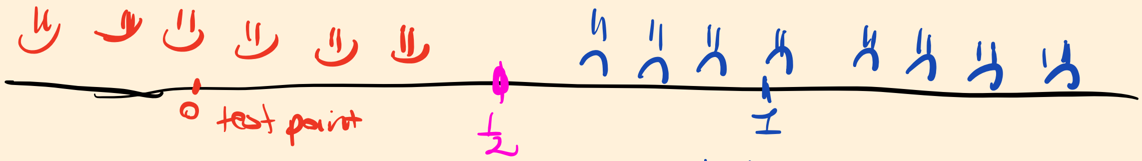
$$0 + 2x = -2x+1+2x$$

$$2x = 1$$

$$\frac{1}{2}\,2x = \frac{1}{2}\,2x$$

$$\boxed{x = \frac{1}{2}}$$

$$0 < -2x+1$$




test point

$-2x+1$
$= -2 \cdot 0 + 1$
$= 0+1$
$= 1$

does satisfy the inequality

$-2x+1 = -2(1)+1 = -2+1 = -1$

does not satisfy the inequality

**Summarizing:**

to solve a linear inequality we solve instead the corresponding equality.
Then we test the inequality for a point to the left and to the right of the solution found

example

$$-2x+1 < 3x+5$$

solve instead

$$-2x+1 = 3x+5$$

$$-2x+1+2x = 3x+5+2x$$

$$1 = 5x+5$$

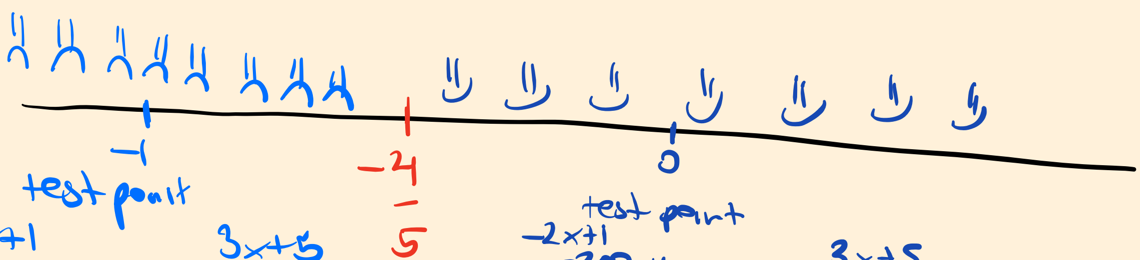
$$1-5 = 5x+5-5$$

$$-4 = 5x$$

$$-4 \cdot \frac{1}{5} = 5x \cdot \frac{1}{5}$$

$$-\frac{4}{5} = x$$

$$-2x+1 < 3x+5$$




test point

$-2x+1$
$= -2(-1)+1$
$= 2+1$
$= 3$

$3x+5$
$= 3(-1)+5$
$= -3+5$
$= 2$

inequality is not satisfied

test point

$-2x+1$
$= -2\cdot 0+1$
$= 0+1$
$= 1$

$3x+5$
$= 3\cdot 0+5$
$= 0+5$
$= 5$

inequality is satisfied

The same method works with linear inequalities in two variables

example

solve

$$0 < y + x - 1$$

we solve the equality first

$$0 = y + x - 1$$

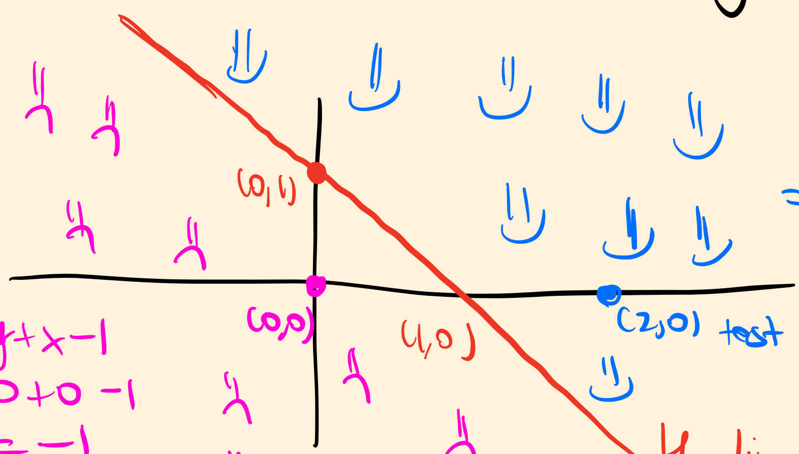
$$0 + 1 = y + x - 1 + 1$$

$$1 = y + x$$

$$1 - x = y + x - x$$

$$\boxed{\begin{array}{c} 1 - x = y \\ y = -x + 1 \end{array}}$$

$$0 < y + x - 1$$




$y + x - 1$
$= 0 + 2 - 1$
$= 1$ satisfies inequality

$y + x - 1$
$= 0 + 0 - 1$
$= -1$
does not satisfy the inequality

the line divides the plane in two regions
we choose a point from each region

# Solving two inequalities : and versus or

For many problems we want to solve more than one inequality at a time.

example

solve $2x+1 \leq 3$ and $-x+4<5$

The and means that we want values of "x" which satisfy both inequalities simultaneously. so we must solve each individually first

$$2x+1 \leq 3$$

$$2x+1=3$$

$$2x+1-1=3-1$$

$$2x=2$$

$$\frac{1}{2}\,2x=\frac{1}{2}\,2$$

$$\boxed{x=1}$$

$$-x+4<5$$

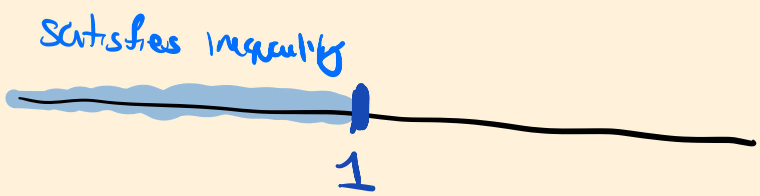
$$-x+4=5$$

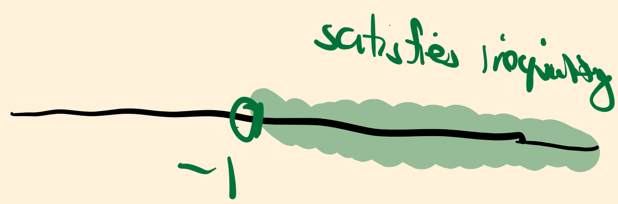
$$-x+4+x=5+x$$

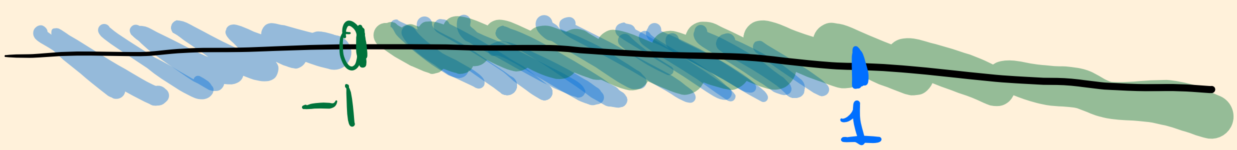
$$4=5+x$$

$$4-5=5+x-5$$

$$\boxed{-1=x}$$



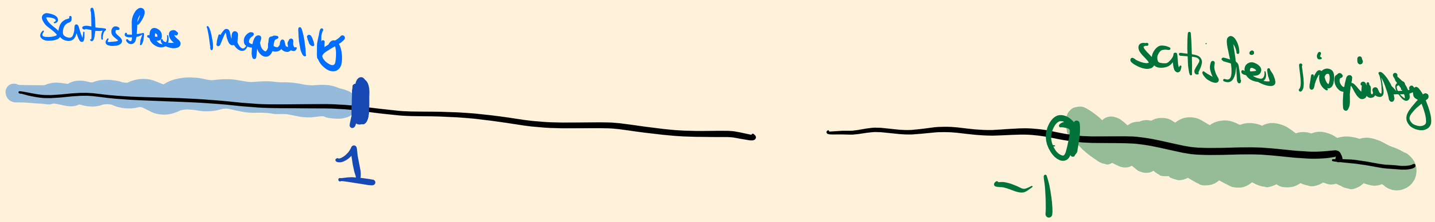

For AND we need the region colored twice so the interval $[-1, 1]$

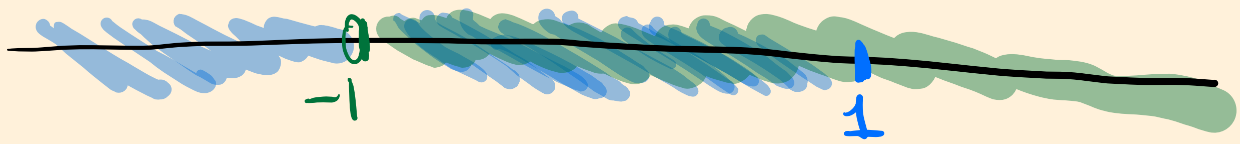
For OR we just want at least one of the inequalities to be satisfied

example

Solve $2x+1 \le 3$ or $-x+4<5$

these inequalities are the same as before, so we find




For or we need the regions colored at least twice, that is, any value on the number line in this case.

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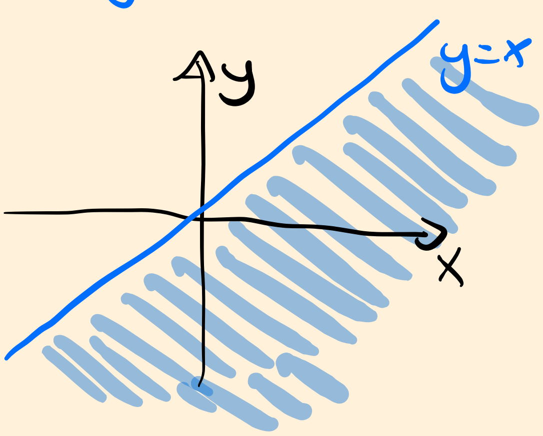
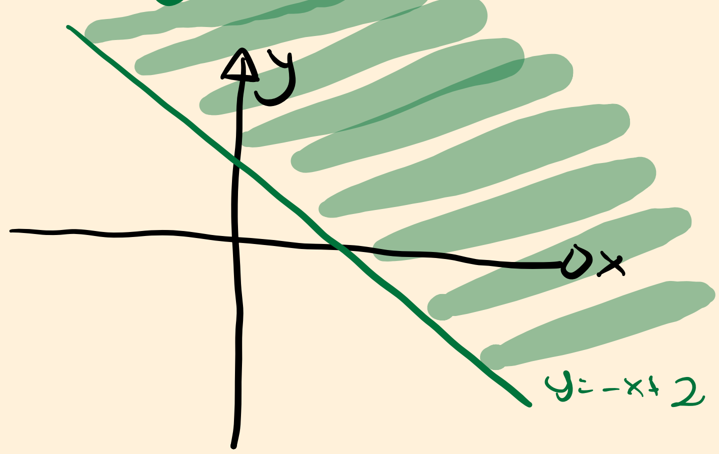
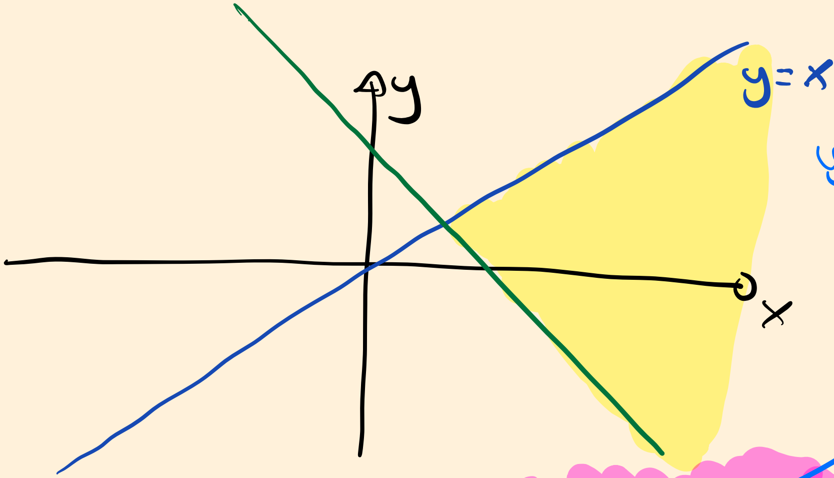
same works for linear inequalities in two variables

$y \le x$ and $y \ge -x+2$

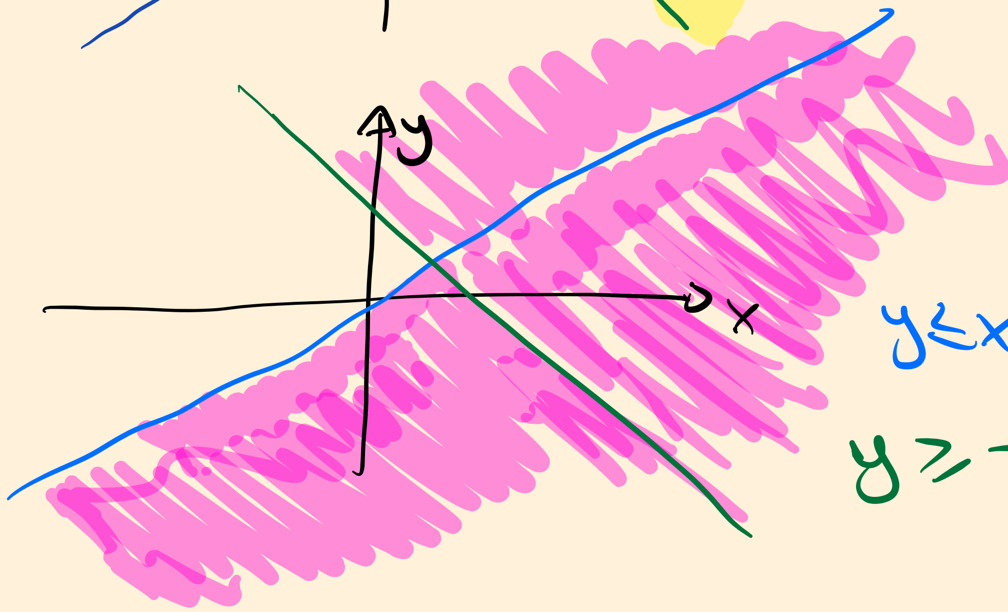





$y \le x$ and

$y \ge -x+2$




$y \le x$ or

$y \ge -x+2$

## Part 2: 2.5, 2.6

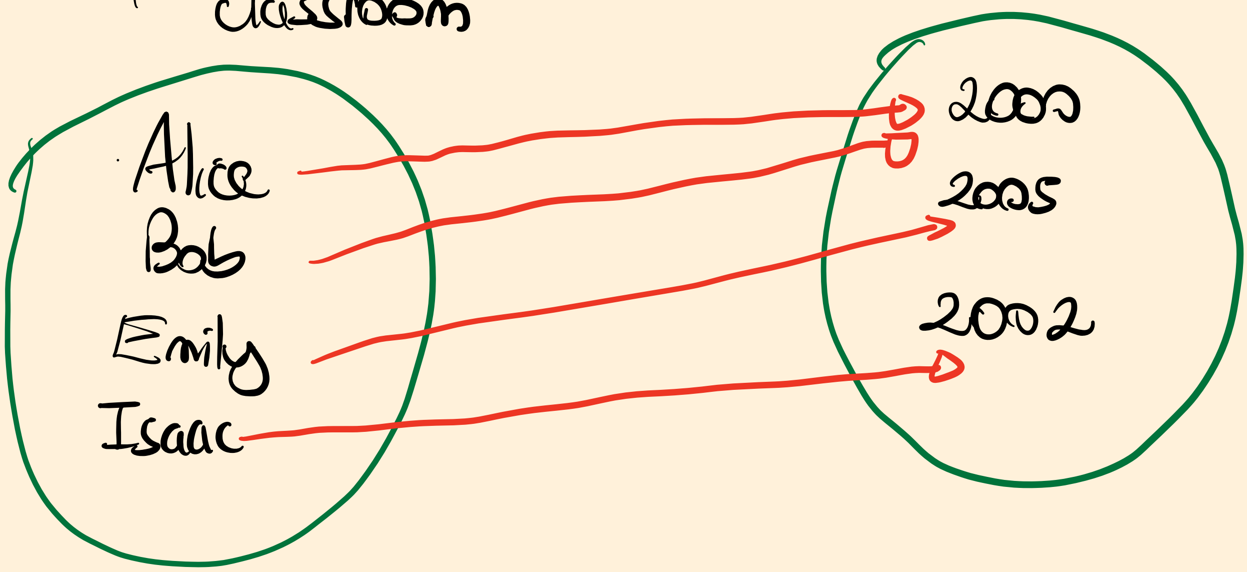
A function is a relation between two quantities or objects.
One of them is called the input while the other one is called the output.

The main property is that
for a function every possible input value is assigned a single output value

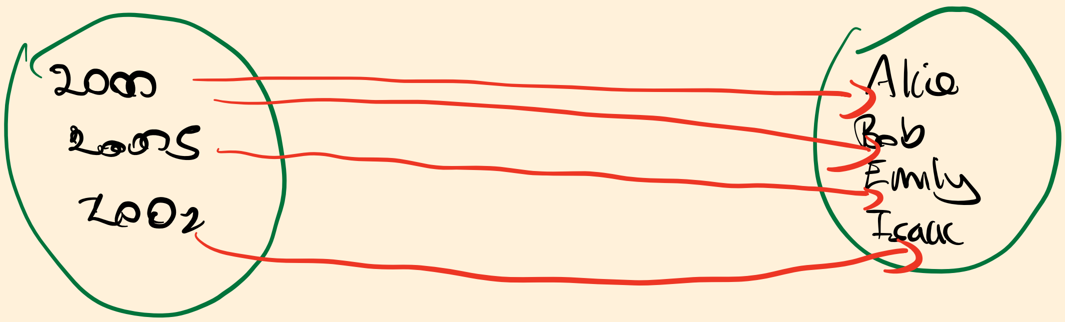
### example 1

input = people in the classroom

output = year of birth




Notice that we can't turn things around.
For example we can't have



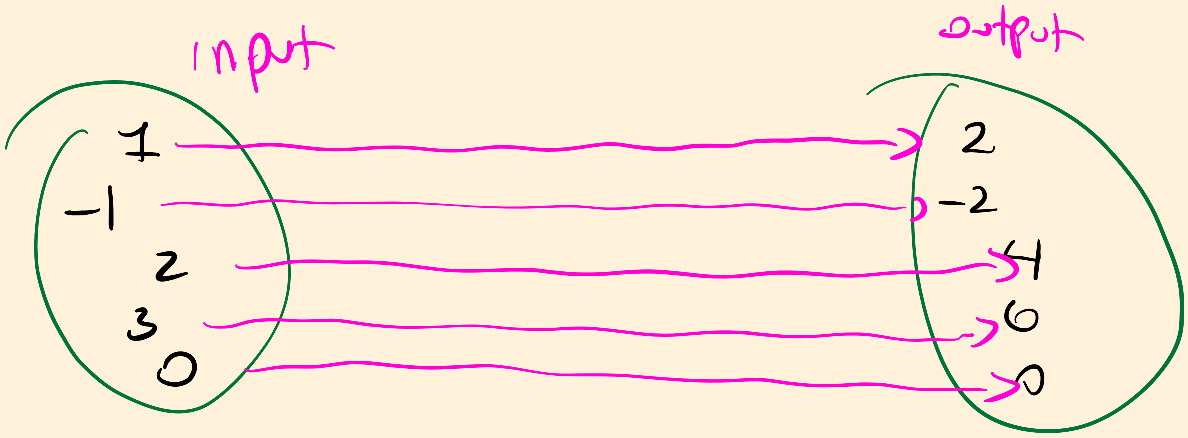

In this case one value of the input (2000) is being assigned two output values, which is not allowed.

In general we are interested in functions where the both the input and output are represented by numbers on the number line

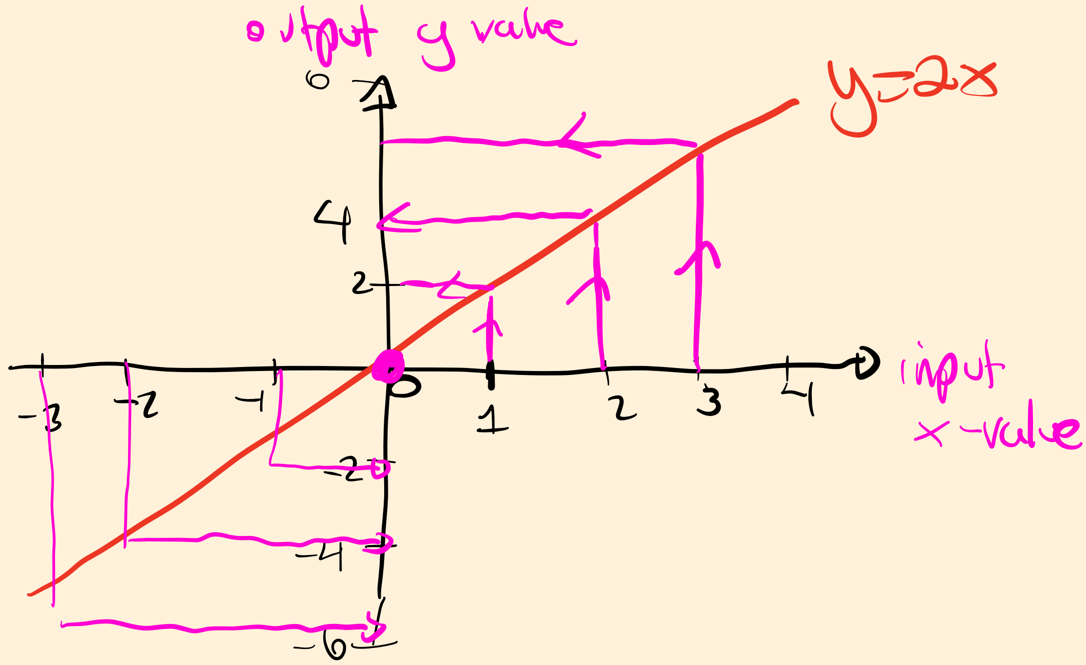
example:

input: any number x

output: twice the value of x



However in this case a better representation is using the number line



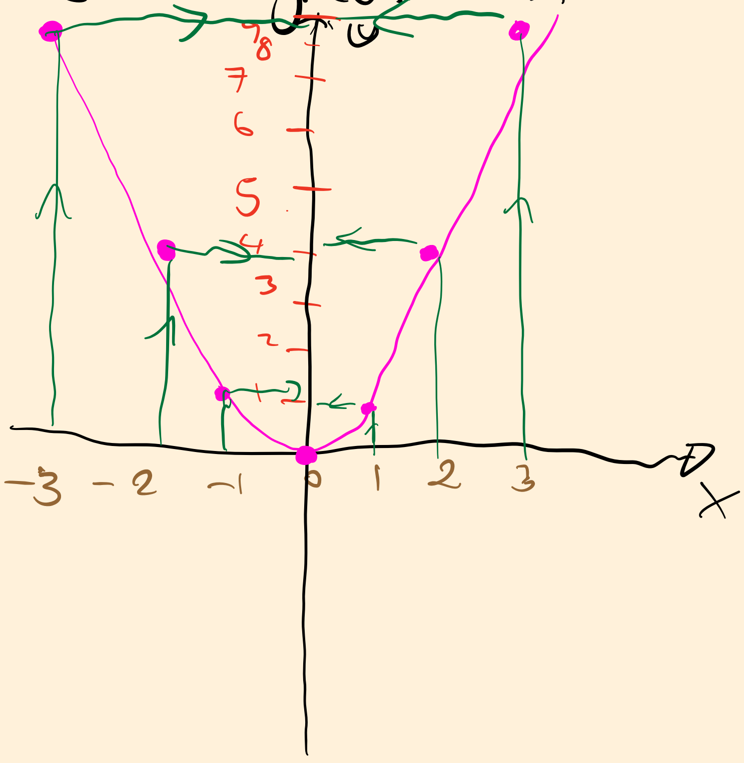

In general, we write $y = f(x)$ to refer to a function

$f(x)$ is the value that is assigned to "x"

example:

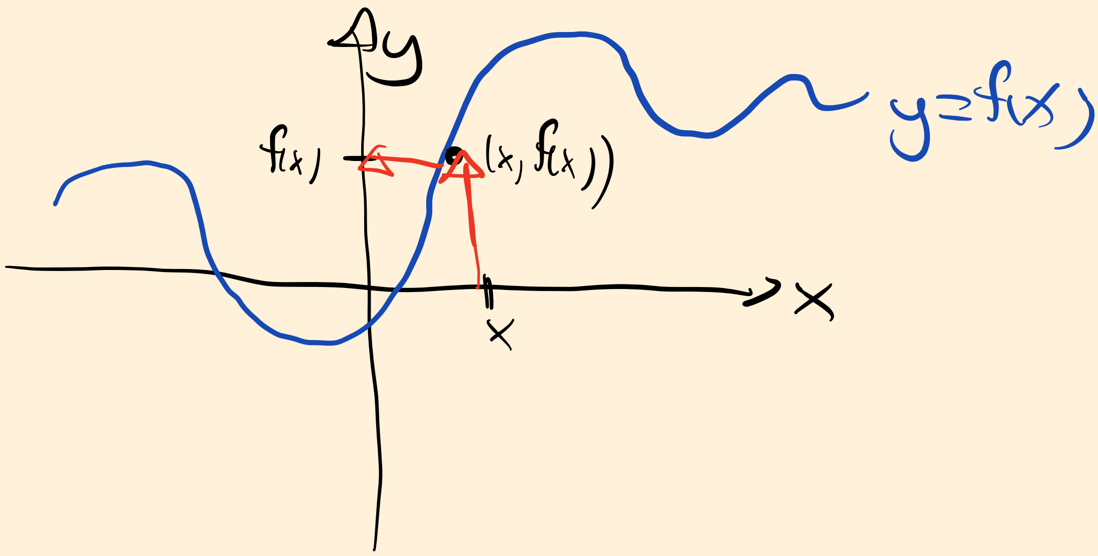
$f(x) = x^2$

| $x$ | $f(x) = x^2$ |
|---|---|
| 0 | $f(0) = 0^2 = 0$ |
| 1 | $f(1) = 1^2 = 1$ |
| 2 | $f(2) = 2^2 = 4$ |
| 3 | $f(3) = 3^2 = 9$ |
| -1 | $f(-1) = (-1)^2 = 1$ |
| -2 | $f(-2) = (-2)^2 = 4$ |

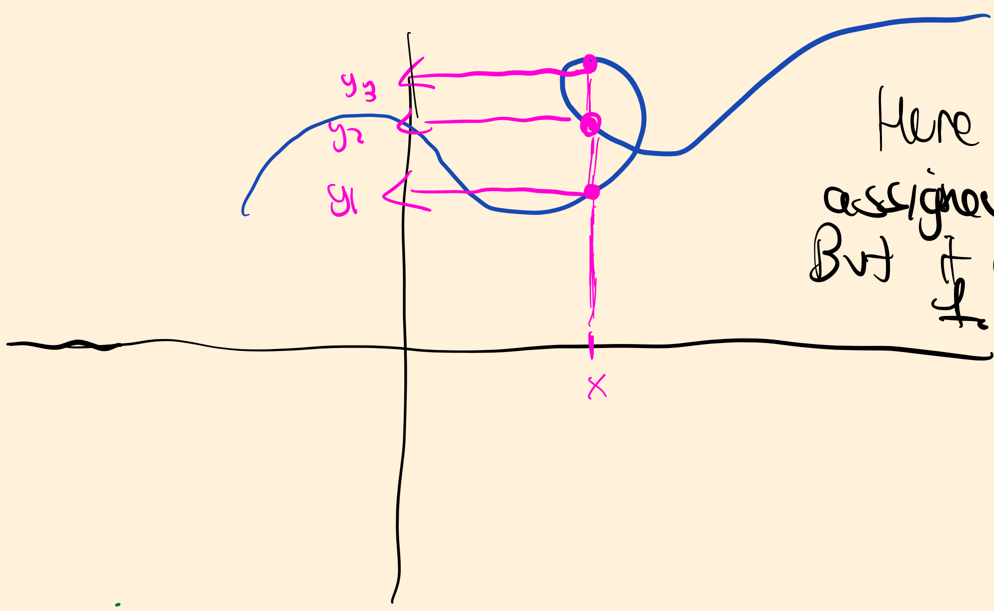


vertical line test

In general, the graph of a function $y=f(x)$ will correspond to a curve on the xy plane




However, not every curve can correspond to the graph of a function



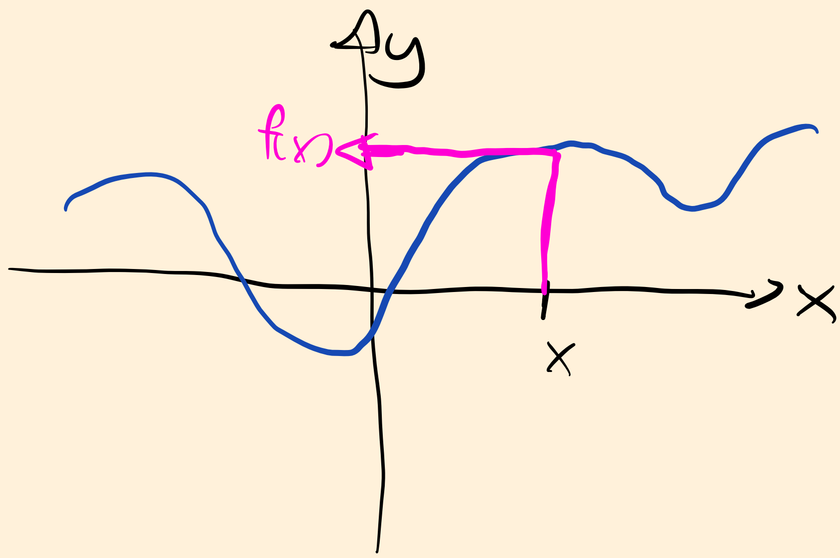

Here x is being assigned 3 values
But it can only be assigned 1

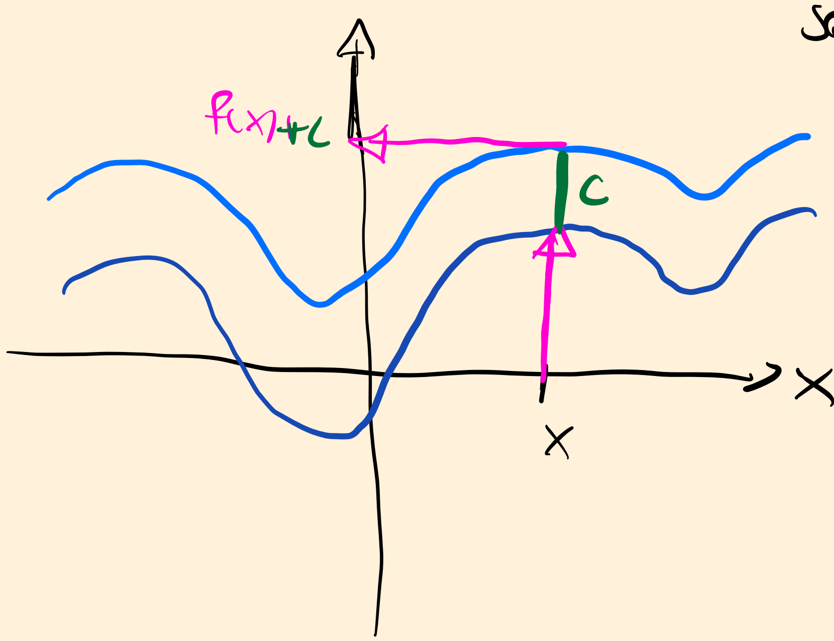
vertical line test

If a curve on the xy plane intersects at least one vertical line at more than one point, then this curve does not represent the graph of a function.

Shifting operations

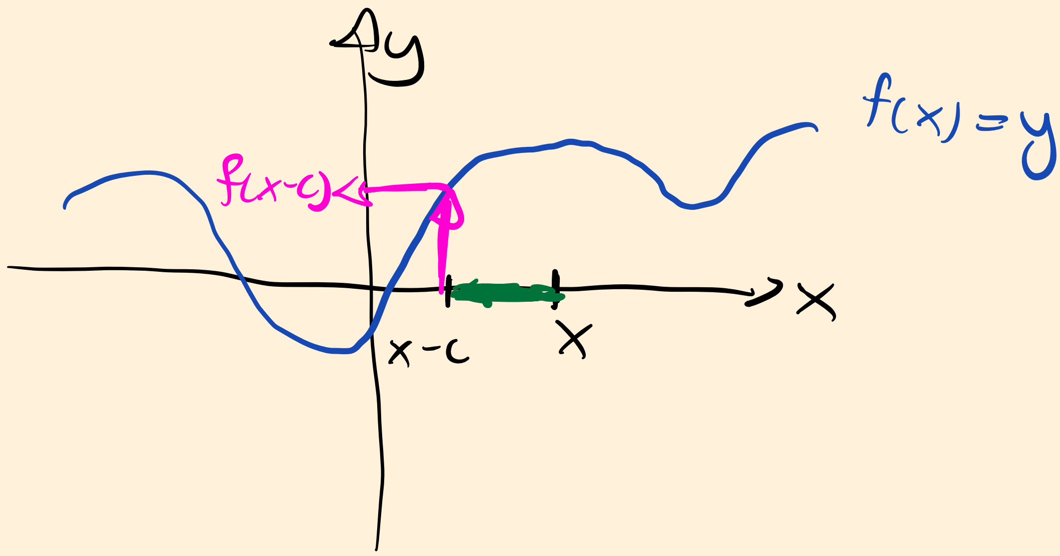
$g(x) = f(x) + c$

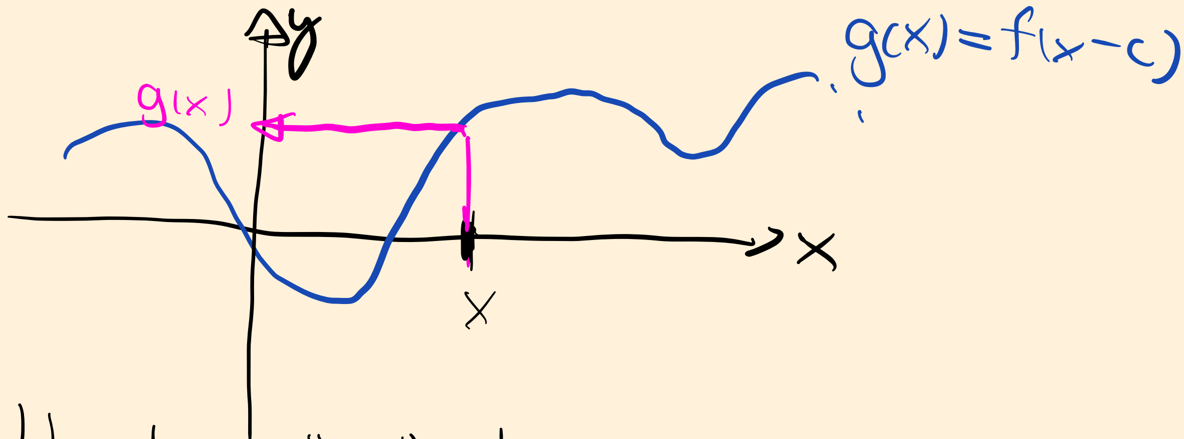



so adding "c" translates the graph vertically by "c" units

$g(x) = f(x-c)$




so subtracting "c" to the input moves the graph to the right by "c" units

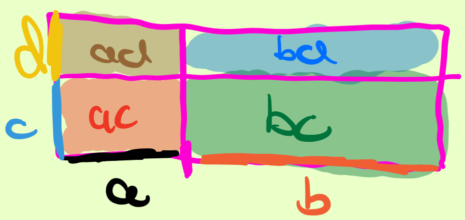
Projects: Multiplication and Factorization

sections (5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9)

Distributive Property (key identity)

$$(a+b)(c+d) = ac + ad + bc + bd$$

idea behind it: this product represents the area of a square with sides of lengths a+b, c+d




In practice we read the previous expression from right to left

$$ac + ad + bc + bd$$

$$= a(c+d) + b(c+d)$$

$$= (a+b)(c+d)$$

In other words, one can group the terms of the previous expression in groups factoring out the common quantities from the different terms

This works even when there are apparently more or less than 4 terms

examples:

Factor

$$x^2+5x+6$$

$$= x^2+2x+3x+6$$

$$= x(x+2)+3(x+2)$$

$$= (x+2)(x+3)$$

In other words $x^2+5x+6=(x+2)(x+3)$

why is this useful: suppose you wanted to solve

$$x^2+5x+6=0$$

This is hard to do directly, but if you factor $x^2+5x+6$, then we need to solve instead

$$(x+2)(x+3)=0$$

Then we use

If the product of two numbers is 0, then at least one of them must be 0

Hence we end up with

$x+2=0$ or $x+3=0$

$x=-2$ or $x=-3$

In general we want to consider an expression like

$$x^2 + bx + c$$

we want to write it as a product

$$(x + d_1)(x + d_2)$$

where we must find the values for $d_1, d_2$

Using the distributive property

$$(x+d_1)(x+d_2)$$

$$= x^2 + xd_2 + d_1 x + d_1 d_2$$

$$= x^2 + (d_1 + d_2)x + d_1 d_2$$

comparing with

$$x^2 + bx + c$$

we want

$$d_1 + d_2 = b$$

$$d_1 d_2 = c$$

In other words, to factorize

$$x^2 + bx + c$$

we find two numbers whose product is the constant term, and who sum is the coefficient of $x$.

ex 1

Factor

$$x^2 + 14x + 48$$

try a few options

$48 = 1 \cdot 48$

$48 = 2 \cdot 24$

$48 = 4 \cdot 12$

$48 = 6 \cdot 8$

the factors of the last option add to 14 so we write

$$x^2 + 14x + 48 = (x+6)(x+8)$$

ex2:

Factor

$$x^2 - 16x + 55$$

we first rewrite this as

$$x^2 + (-16x) + 55$$

so when we decompose 55 as a product of two numbers, we write it as the product of two negative numbers

$55 = (-1)(-55)$

$55 = (-5)(-11)$

since $-5$ and $-11$ add to $-16$ we have

$$x^2 - 16x + 55 = (x + (-5))(x + (-11))$$
$$= (x-5)(x-11)$$

ex3

Factor

$$x^2+3x-4$$

Now we must decompose $-4$ as a product of a positive and a negative number

$$-4 = 1\cdot(-4)$$
$$-4 = 2\cdot(-2)$$
$$-4 = 4\cdot(-1)$$

4 and $-1$ add up to 3 so we have

$$x^2+3x-4 = (x+4)(x+(-1))$$
$$=(x+4)(x-1)$$

ex4

Factor

$$2x^2-3x-9$$

we first rewrite the expression so that the coefficient of $x^2$ is 1.

$$2x^2-x-9 = 2x^2-\frac{2}{2}\cdot 3x-\frac{2}{2}\cdot 9$$

$$=2\left(x^2-\frac{3x}{2}-\frac{9}{2}\right)$$

$\frac{9}{2}=1\cdot\frac{9}{2}$ | $\frac{9}{2}=\frac{3}{2}\cdot 3$ | $\frac{9}{2}=\frac{1}{2}\cdot 9$

$-\frac{9}{2}=(-1)\cdot\frac{9}{2}$ | $-\frac{9}{2}=\left(-\frac{3}{2}\right)\cdot 3$ | $-\frac{9}{2}=\left(-\frac{1}{2}\right)\cdot 9$

$-\frac{9}{2}=(1)\cdot\left(-\frac{9}{2}\right)$ | $-\frac{9}{2}=\frac{3}{2}\cdot(-3)$ | $-\frac{9}{2}=\frac{1}{2}\cdot(-9)$

adds to

$$\frac{3}{2}+-3=\frac{3-6}{2}=\frac{-3}{2}$$

so $x^2 - \frac{3x}{2} - \frac{9}{2} = \left(x+\frac{3}{2}\right)(x-3)$

Hence

$$2x^2 - 3x - 9 = 2\left(x+\frac{3}{2}\right)(x-3)$$
$$= (2x+3)(x-3)$$

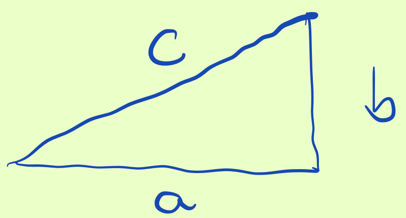
---

# Square roots

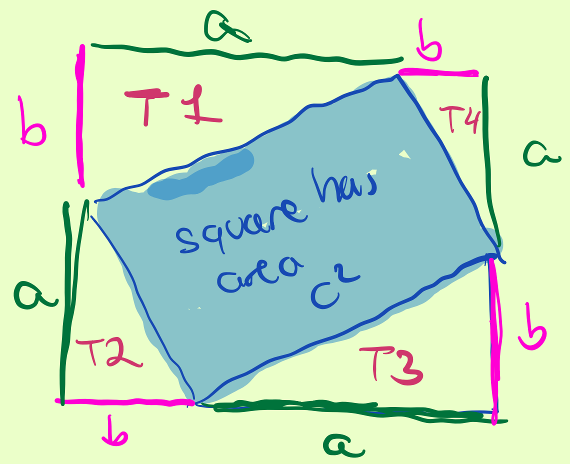
## Pythagora's Theorem

If a, b, c are the sides of a right triangle (c representing the length of the hypothenuse) then

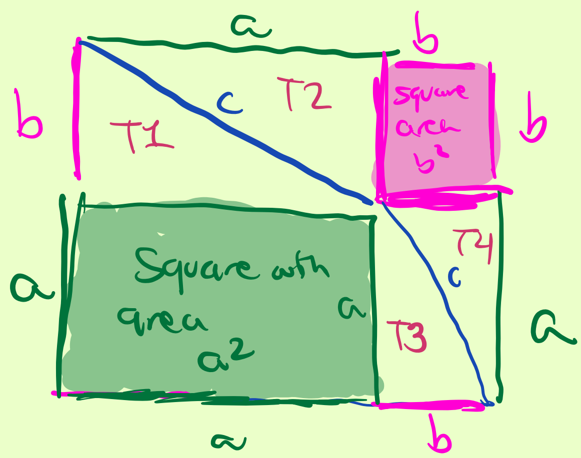
$$c^2 = a^2 + b^2$$




idea




make a square of sides $a+b$

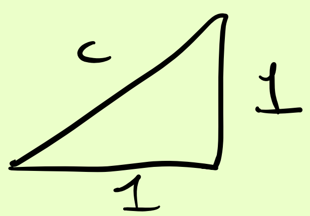



another way to make the same square

since the same four triangles are on each square we must have

$$\boxed{c^2 = a^2 + b^2}$$

Why is this important?



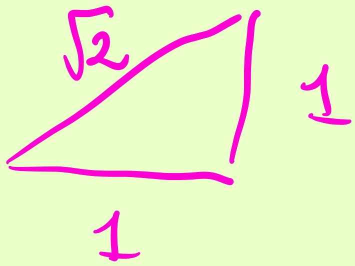

By Pythagoras

$$c^2 = 1^2 + 1^2$$

$$c^2 = 2$$

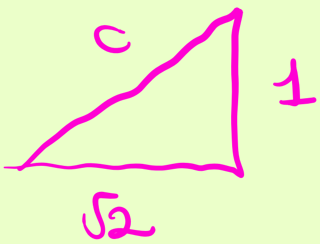
so we have found a number whose square is 2 that is

$$c = \sqrt{2}$$




another triangle

By Pythagoras




$$c^2 = (\sqrt{2})^2 + 1^2$$

$$c^2 = 2 + 1$$

$$c^2 = 3$$

so $\qquad c = \sqrt{3}$

And that is how one can find $\sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}$, etc

Notation:

$\sqrt{n}$ means a number whose square is $n$.

$$\boxed{(\sqrt{n})^2 = n.}$$

Remarks:

This requires "n" non-negative that is $\sqrt{5}$ makes sense but $\sqrt{-5}$ doesn't (because we don't use imaginary numbers, which do allow to make sense of $\sqrt{-5}$)

## Second remarks

Take $n=4$ as an example. Then $\sqrt{4}=2$ since $2^2=4$. However, we also have $(-2)^2=4$ which means that 4 has two square roots, which are $+2$ and $-2$

In general, if $n$ is positive, then $n$ has two square roots $\sqrt{n}$ and $-\sqrt{n}$

## Solving quadratic equations

We want to solve

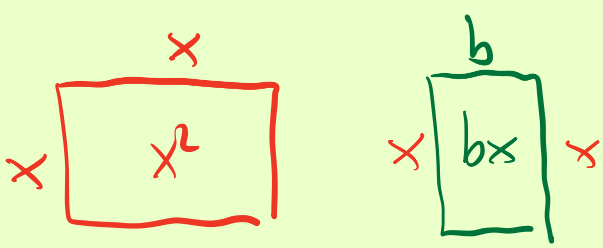
$$x^2+bx+c=0$$

We rewrite the equation as

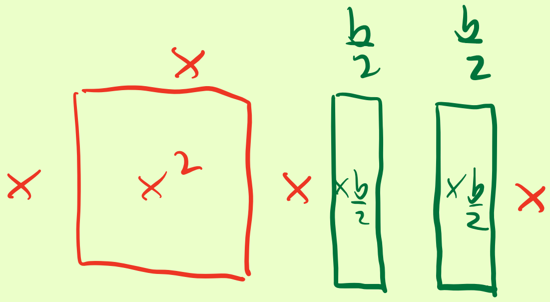
$$x^2+bx=-c$$

and reinterpret the left hand side geometrically

$$x^2+bx=-c$$

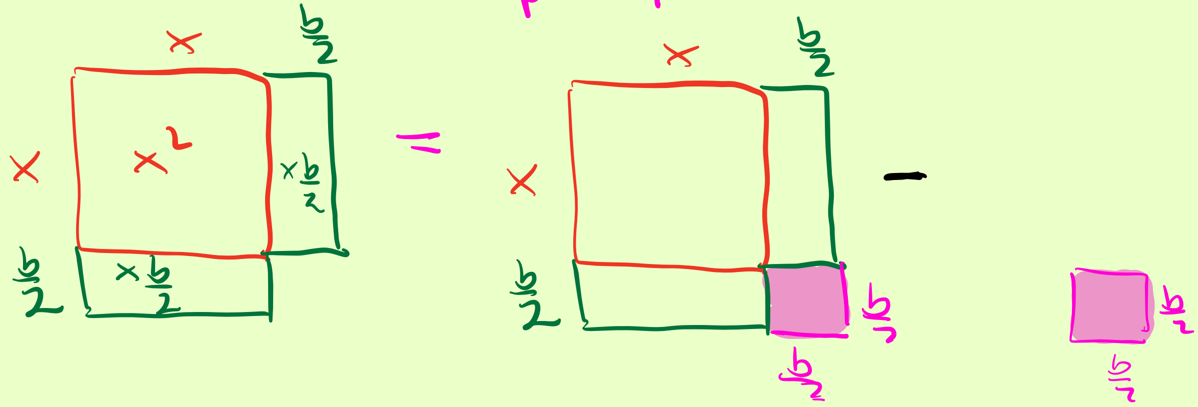



We divide the second rectangle into two pieces




rearrange rectangle

complete square




$$x^2 + bx = \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4}$$

so the equation

$$x^2 + bx = -c$$

becomes

$$\left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} = -c$$

or

$$\left(x+\frac{b}{2}\right)^2 = \frac{b^2}{4} - c$$

Hence

$x+\frac{b}{2}$ is a number whose square is $\frac{b^2}{4}+c$

in other words

$$x+\frac{b}{2}=\sqrt{\frac{b^2}{4}-c} \quad \text{or} \quad x+\frac{b}{2}=-\sqrt{\frac{b^2}{4}-c}$$

So

$$\boxed{x=-\frac{b}{2}+\sqrt{\frac{b^2}{4}-c}} \quad \text{or} \quad \boxed{x=-\frac{b}{2}-\sqrt{\frac{b^2}{4}-c}}$$

example:

Solve

$$x^2+4x+1=0$$

we complete square (here $b=4$)

$$x^2+4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+1=0$$

$$\underbrace{x^2+4x+4}-4+1=0$$

$$(x+2)^2-3=0$$

$$(x+2)^2=3$$

$$x+2=\sqrt{3} \quad \text{or} \quad x+2=-\sqrt{3}$$

$$x=-2+\sqrt{3} \quad \text{or} \quad x=-2-\sqrt{3}$$

Example:

Solve

$$2x^2 - 3x - 1 = 0$$

multiply by $\frac{1}{2}$

$$\frac{1}{2}\left(2x^2 - 3x - 1\right) = \frac{1}{2} \cdot 0$$

$$x^2 - \frac{3}{2}x - \frac{1}{2} = 0$$

Here $b = -\frac{3}{2}$

$$x^2 - \frac{3}{2}x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 - \frac{1}{2} = 0$$

$$x^2 - \frac{3}{2}x + \frac{9}{4} - \frac{9}{4} - \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{9}{4} + \frac{1}{2}$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{11}{4}$$

$$x - \frac{3}{4} = \sqrt{\frac{11}{4}} \quad \text{or} \quad x - \frac{3}{4} = -\sqrt{\frac{11}{4}}$$

$$x = \frac{3}{4} + \sqrt{\frac{11}{4}} \quad \text{or} \quad x = \frac{3}{4} - \sqrt{\frac{11}{4}}$$

Project 6

# Quadratic Functions

Up to now we studied the linear functions

$$y = f(x) = mx + b$$

In terms of difficulty, quadratic functions are the succesors of linear functions in terms of difficulty

$$y = f(x) = ax^2 + bx + c$$

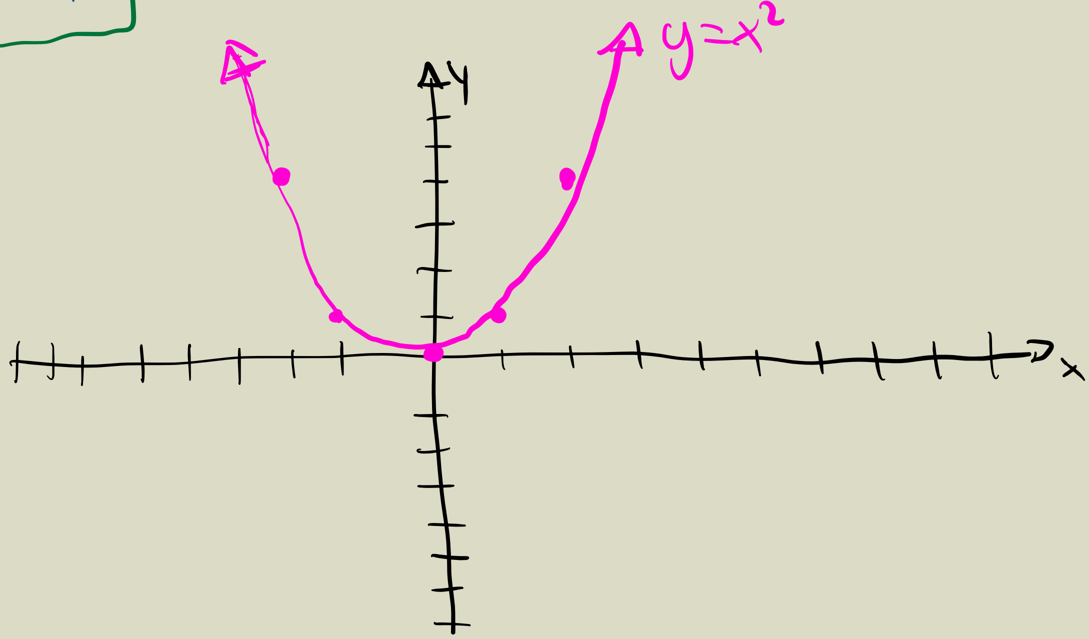
Here a, b, c are numbers to be determined.

We will analyze the graphs in steps:

case $b = c = 0$

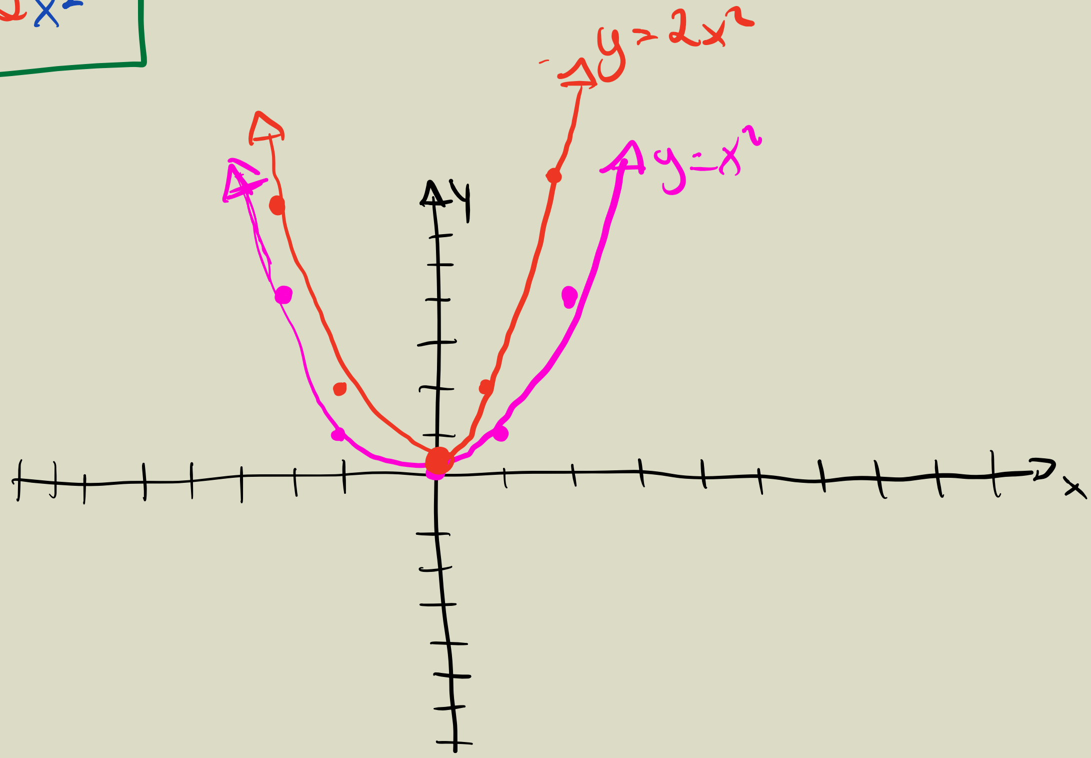
$$\boxed{y = x^2}$$

| $x$ | $y = x^2$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| -1 | 1 |
| 2 | 4 |
| -2 | 4 |



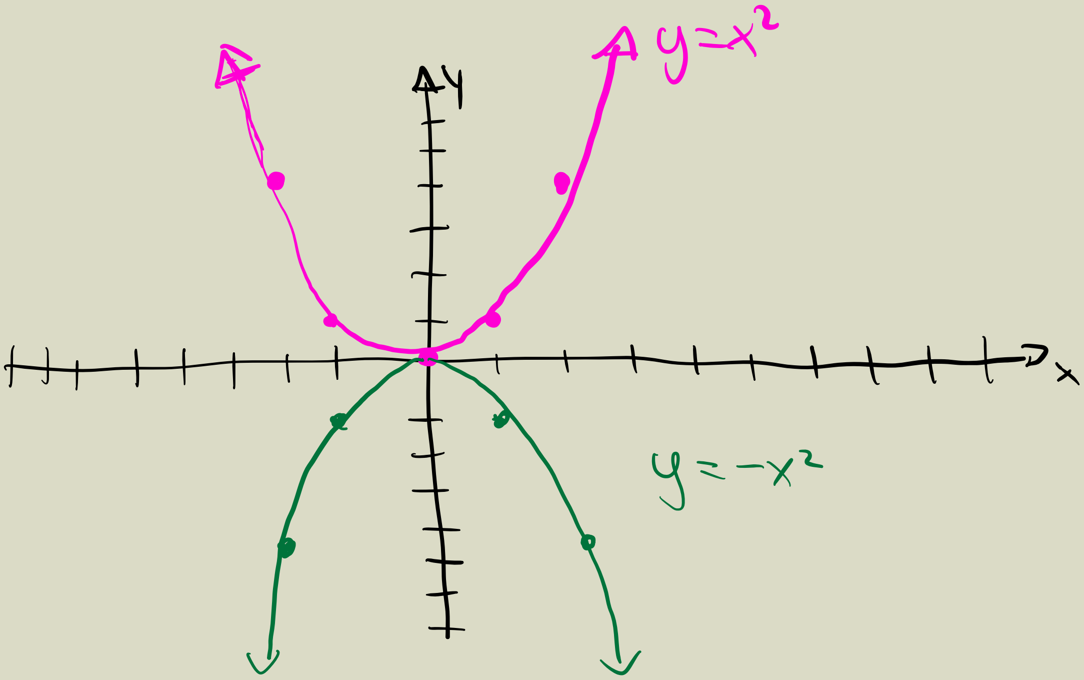
$\boxed{y = 2x^2}$

| $x$ | $y = 2x^2$ |
|---|---|
| 0 | 0 |
| 1 | 2 |
| -1 | 2 |
| 2 | 8 |
| -2 | 8 |




For $y = ax^2$ if $a > 1$, the graph (parabola) is closer to the y-axis than $y = x^2$.

$\boxed{y = -x^2}$



$\boxed{\text{case } c=0}$

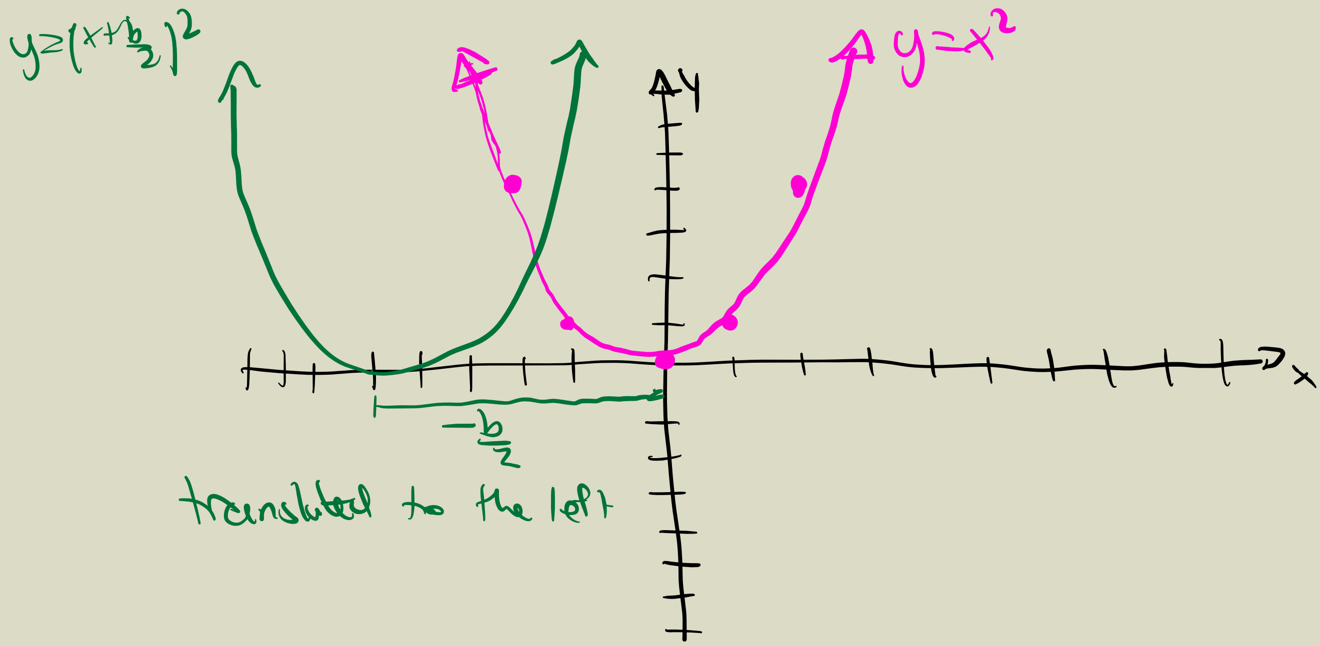
$y = x^2 + bx$

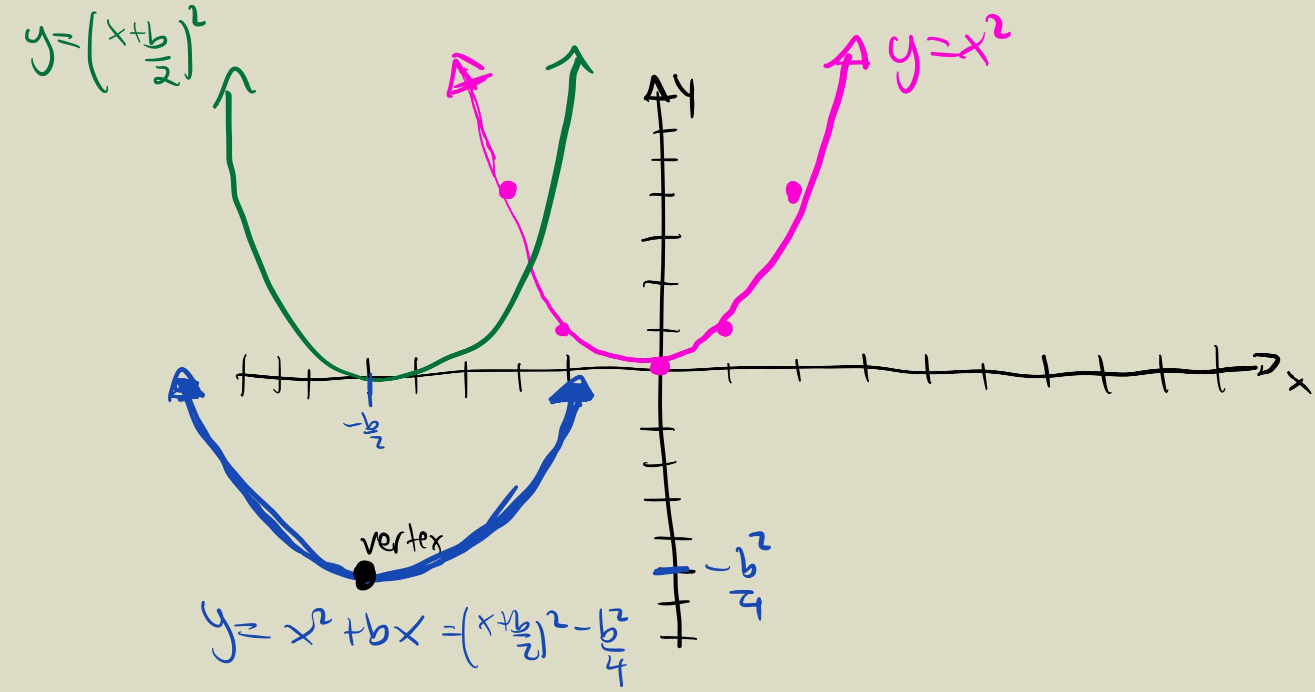
to understand how the graph looks, we complete the squares

$$x^2 + bx = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$$

So

$$y = x^2 + bx = \underbrace{\left( x + \frac{b}{2}\right)^2}_{\text{translates the graph to the left by an amount } \frac{b}{2}} \overbrace{- \frac{b^2}{4}}^{\text{translate the graph down by } \frac{b^2}{4}}$$



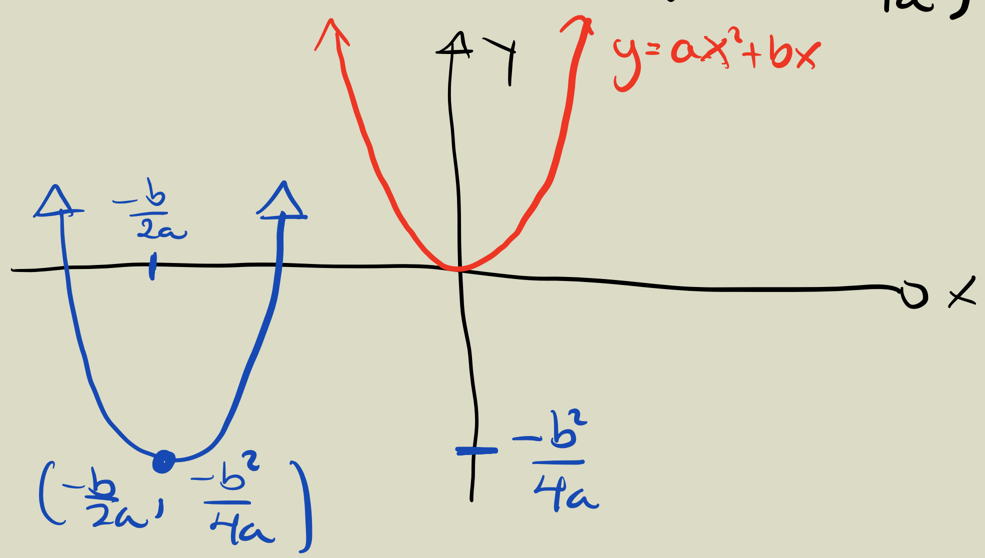



The vertex of the parabola is the lowest (or highest) point of the parabola.

For $y=ax^2+bx$, we use

$$y=ax^2+bx=a\left(x^2+\frac{b}{a}x\right)$$

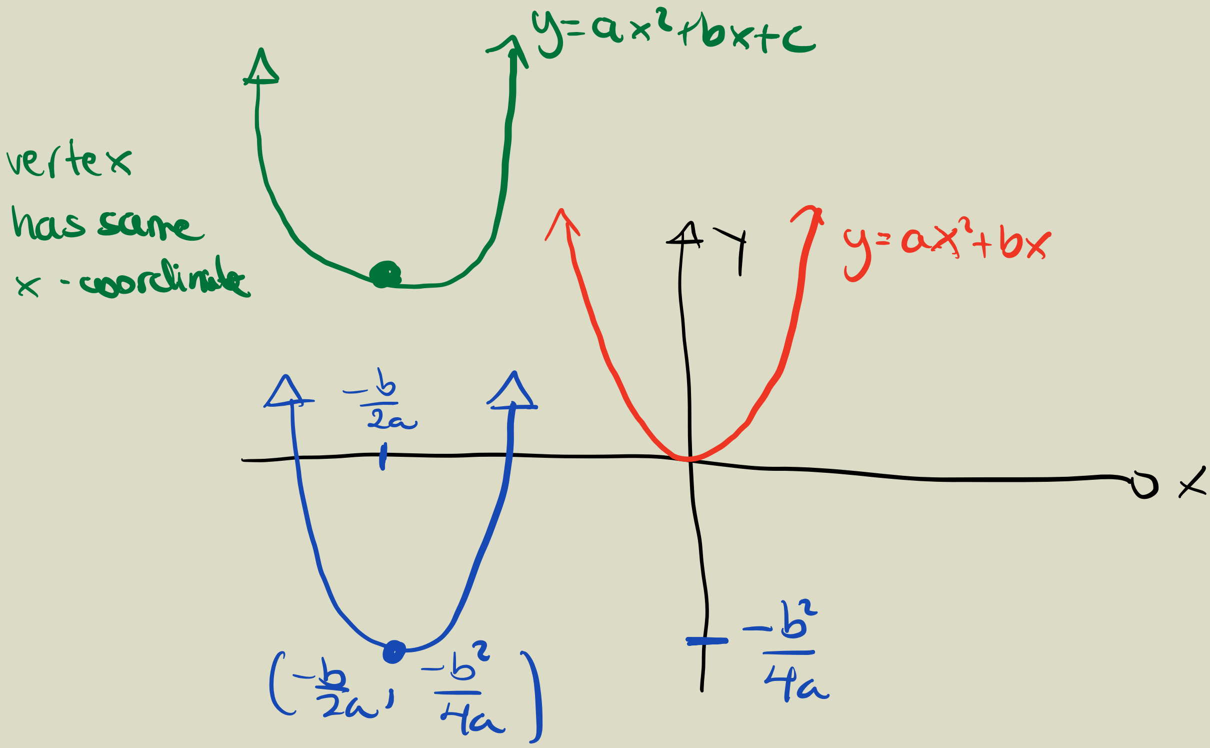
To find the graph of $y=ax^2+bx$ from $y=ax^2$ shift the graph of $y=ax^2$ so that the vertex is the point $\left(\frac{-b}{2a},\frac{-b^2}{4a}\right)$



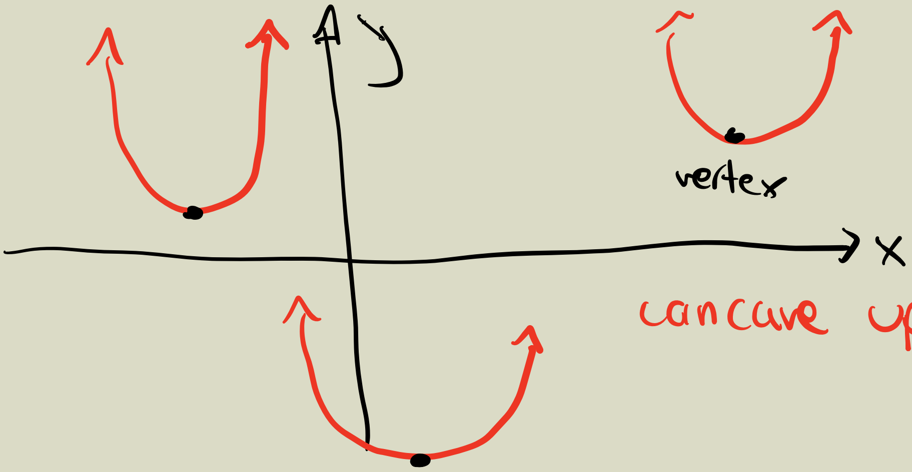
## General case

$$y = ax^2 + bx + c = ax^2 + bx + c$$

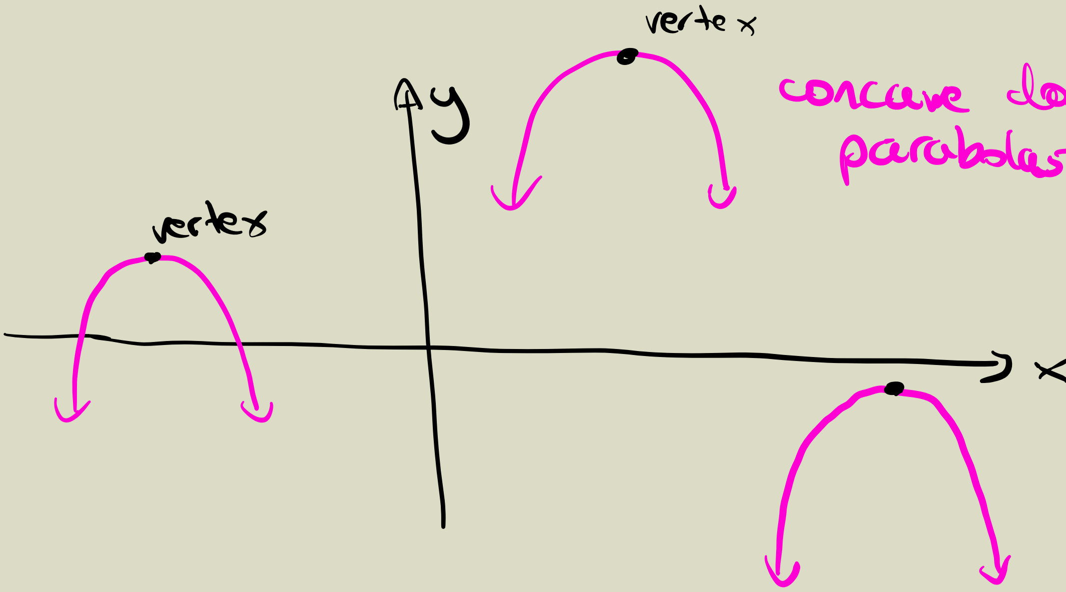
= translates the graph of $y = ax^2 + bx$ vertically by an amount $c$




The function
$$y = f(x) = ax^2 + bx + c \text{ is called}$$
concave up if $a > 0$. It is called concave down if $a < 0$

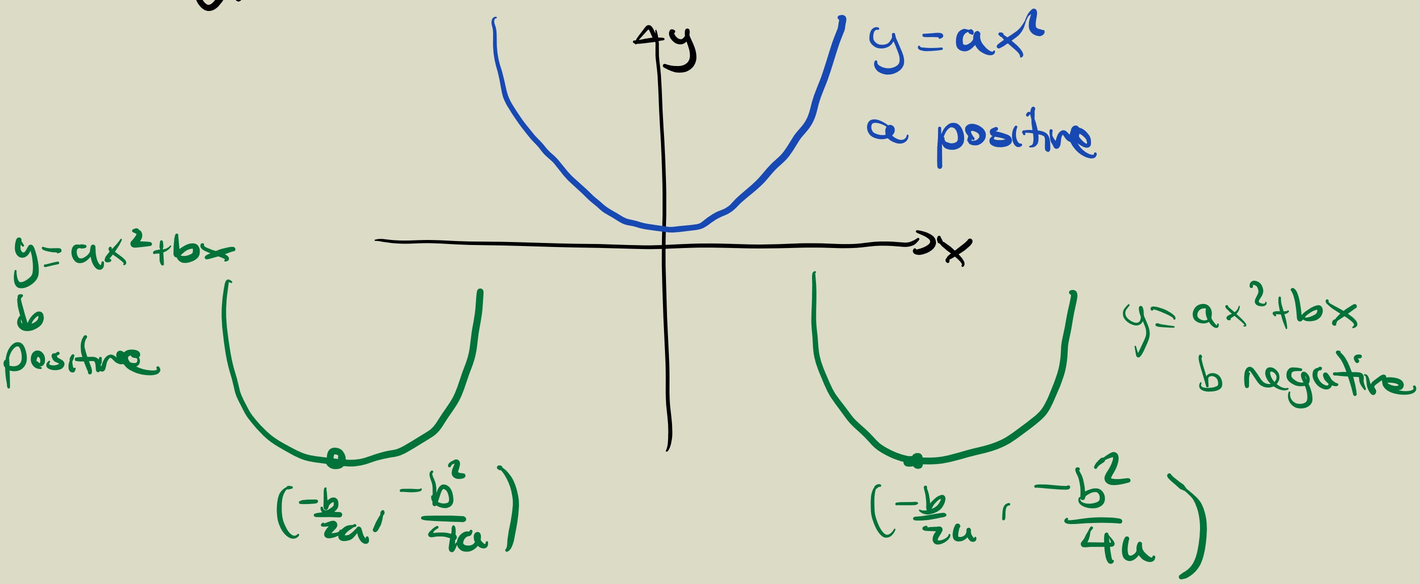


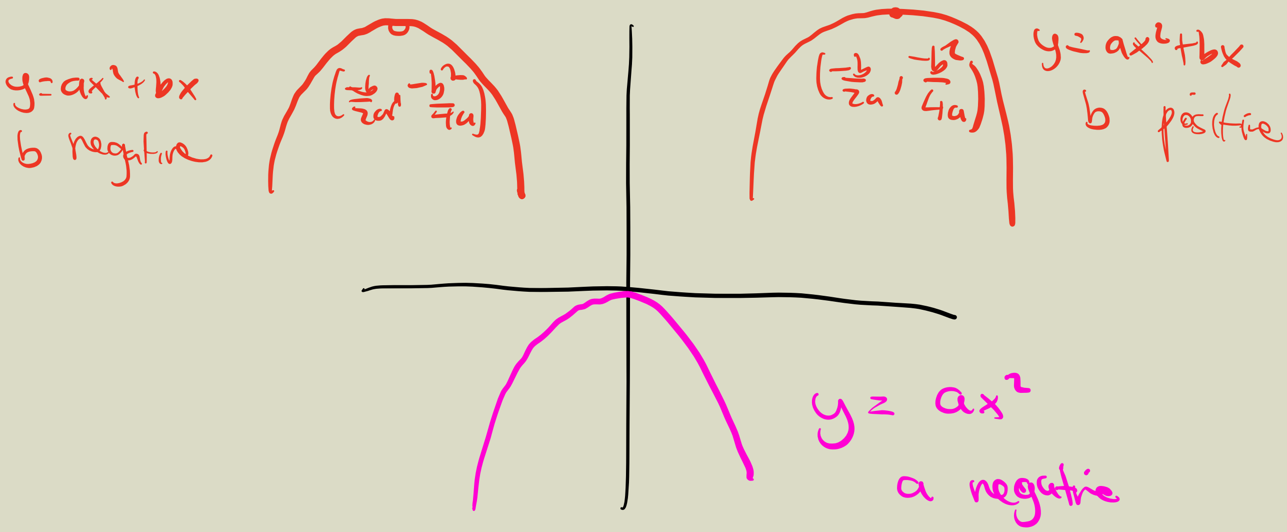

concave up parabolas




concave down parabolas

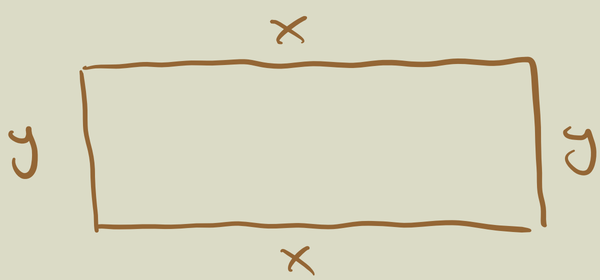
Summary:






## Applications with quadratic equations

Sam wants to build a garden which is shaped as a rectangle
He has 400 ft worth of fencing material.
What must be the dimensions of the terrain if we want it to have the largest amount of area possible?




The area is

$$A = xy$$

and the fencing material is

$$2x + 2y$$

Sam has 400 ft worth of fencing material

So

$$2x+2y=400$$

$$\frac{1}{2}(2x+2y)=\frac{1}{2}\cdot 400$$

$$x+y=200$$

$$x+y-x=200-x$$

$$\boxed{y=200-x}$$

we substitute this into the formula for A

$$A=x(200-x)=200x-x^2=-x^2+200x$$

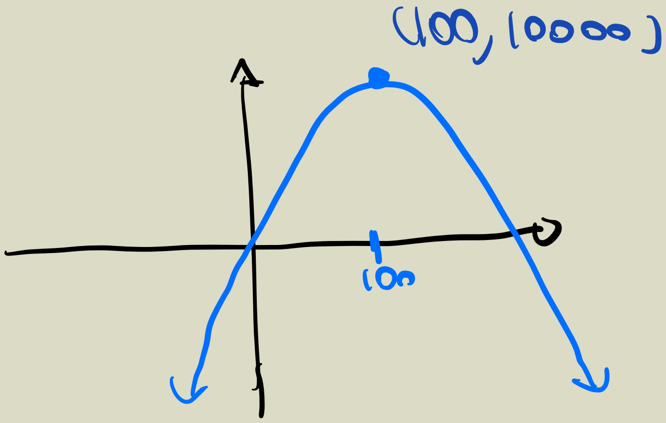
$$A=-x^2+200x$$

↳ concave down parabola

with $a=-1$, $b=200$

vertex of the parabola is at

$$x=\frac{-b}{2a}=\frac{-(200)}{2(-1)}=\frac{-200}{-2}=\boxed{100}$$

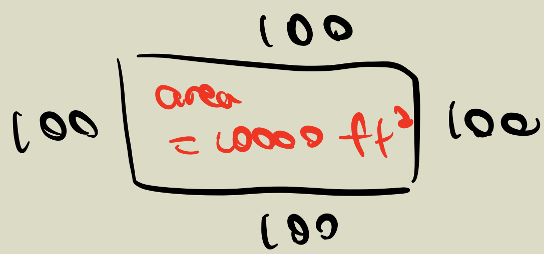



$$A(100)=-(100)^2+200\cdot(100)$$
$$=-10000+20000$$
$$=10000$$

so the value for $x$ is 100 ft,

$$y = 200 - 100 = 100 \text{ ft}$$

so $y$ is also 100 ft,




square terrain

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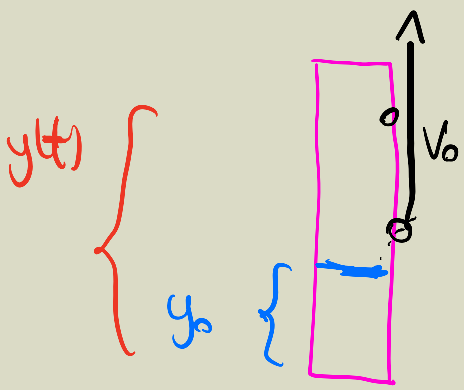
## Projectile motion




Suppose you throw a rock vertically from a floor on a building at height $y_0$.

The height of the projectile at time $t$ is given by

$$y(t) = y_0 + v_0 t - \frac{1}{2} g t^2$$

where $v_0$ is the initial vertical velocity, and $g$ is the acceleration due to gravity, which is a positive constant (roughly $32 \frac{\text{ft}}{\text{sec}^2}$ near the surface of the Earth)

a) what is the maximum height reached by the rock?

b) at what time does the rock hit the ground?

(a) Notice that the formula for "y" is the equation of a parabola, with "t" serving the role of "x"

$$y = -\frac{1}{2}g t^2 + v_0 t + y_0$$

so $a = -\frac{1}{2}g$ $b = v_0$ $c = y_0$

The vertex happens at

$$\boxed{t} = -\frac{b}{2a} = \frac{-v_0}{2\left(-\frac{1}{2}g\right)} = \frac{-v_0}{-g} = \boxed{\frac{v_0}{g}}$$

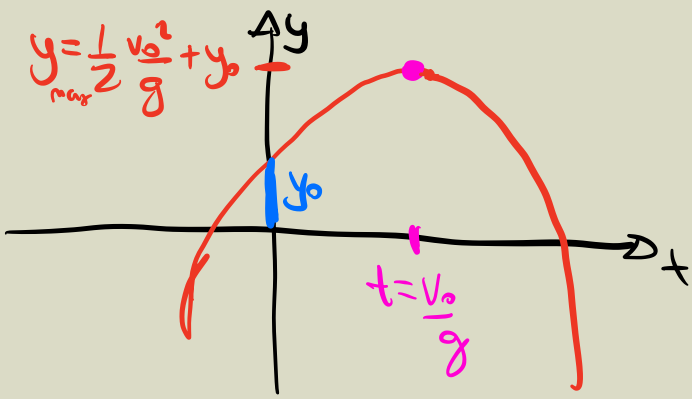
The corresponding value of $y$ for this value of "t" is

$$y = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + y_0$$

$$y = -\frac{1}{2}g\,\frac{v_0^2}{g^2} + \frac{v_0^2}{g} + y_0$$

$$y = -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} + y_0$$

$$y_{max} = \frac{1}{2}\frac{v_0^2}{g} + y_0$$



ⓑ The rock hits the ground when its height is 0, that is when

$$0 = -\frac{1}{2}g t^2 + v_0 t + y_0$$

$$-\frac{2}{g} \cdot \quad 0 = -\frac{2}{g}\left(-\frac{1}{2}g t^2 + v_0 t + y_0\right)$$

$$0 = t^2 \underbrace{- \frac{2v_0}{g}}_{b} t - \frac{2}{g} y_0$$

complete squares with $b = -\frac{2v_0}{g}$, $\frac{b}{2} = -\frac{v_0}{g}$

$$0 = \left(t + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 - \frac{2}{g} y_0$$

$$0 = \left(t - \frac{v_0}{g}\right)^2 - \left(-\frac{v_0}{g}\right)^2 - \frac{2}{g} y_0$$

$$0 = \left(t - \frac{v_0}{g}\right)^2 - \frac{v_0^2}{g^2} - \frac{2}{g} y_0$$

$$\frac{v_0^2}{g^2} + \frac{2}{g} y_0 = \left(t - \frac{v_0}{g}\right)^2$$

so $$\pm\sqrt{\frac{v_0^2}{g^2} + \frac{2y_0}{g}} = t - \frac{v_0}{g}$$

$$\boxed{t = \frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2y}{g}}}$$

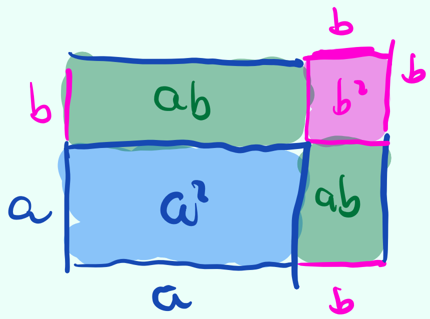
Project 7

## More about Factorization and Radicals

### Special Products

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

remark!
notice that $(a+b)^2$ does not equal $a^2+b^2$




we can find this formula algebraically using the distributive property

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2$$

$$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

this can be proved geometrically in terms of volumes

Alternatively we use the distributive property

$$(a+b)^3 = (a+b)(a+b)^2$$
$$= (a+b)(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

The formula for $(a+b)^n$ can be found with the help of Pascal's Triangle

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3a^2b^{+}1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3\,1b^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

---

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

This can be found geometrically or using the distributive formula.

Alternatively we write

$$(a-b)^2 = (a+(-b))^2$$
$$= a^2 + 2a(-b) + (-b)^2$$
$$= a^2 - 2ab + b^2$$

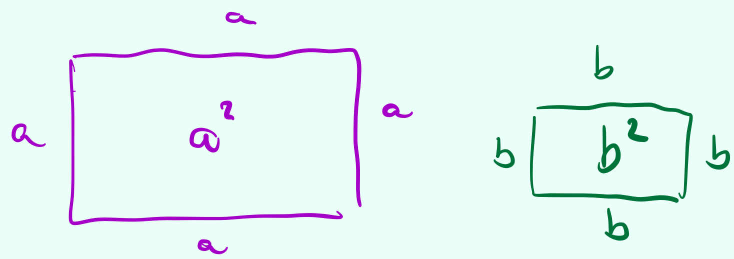
$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

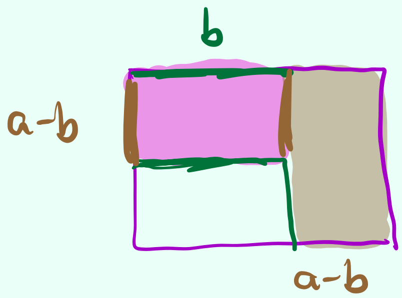
same idea as before

$$(a-b)^3 = (a+(-b))^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3$$

# Difference of squares

$$\boxed{a^2-b^2=(a-b)(a+b)}$$




$$= (a-b)a + (a-b)b$$

$$= (a-b)(a+b)$$

# Difference of cubes

$$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

One way to find this is by drawing to cubes and comparing their volumes. More algebraically

$$a^3-b^3 = a^3-ab^2+ab^2-b^3$$

$$= a(a^2-b^2)+b^2(a-b)$$

$$= a(a-b)(a+b)+b^2(a-b)$$

$$= (a-b)(a(a+b)+b^2)$$

$$= (a-b)(a^2+ab+b^2)$$

$$\boxed{a^3+b^3 = (a+b)(a^2 - ab + b^2)}$$

we use

$$\begin{aligned} a^3+b^3 &= a^3 - (-b)^3 \\ &= (a-(-b))(a^2+a(-b)+(-b)^2) \\ &= (a+b)(a^2-ab+b^2) \end{aligned}$$

example:

Factor

$$x^4 - 16$$

$$\begin{aligned} x^4 - 16 \\ &= (x^2)^2 - (4)^2 \\ &= a^2 - 4^2 \quad \text{where } a = x^2 \\ &= (a-4)(a+4) \\ &= (x^2-4)(x^2+4) \\ &= (x^2-2^2)(x^2+4) \\ &= (x-2)(x+2)(x^2+4) \end{aligned}$$

Factor

$x^2 - 2$

$$x^2 - 2$$

$$= x^2 - (\sqrt{2})^2$$

$$= (x - \sqrt{2})(x + \sqrt{2})$$

Project 8

## Square roots and some properties

Recall that if $a$ is a positive number $\sqrt{a}$ is the number which satisfies

$$(\sqrt{a})^2 = a$$

Notice that $(-\sqrt{a})^2 = a$ as well

so in general a positive number has two square roots, $\sqrt{a}$ and $-\sqrt{a}$, and when we write $\sqrt{a}$ to refer to the positive square root.

### Some properties

$$\sqrt{0} = 0$$

$\sqrt{a^2} = |a|$ since $\sqrt{\text{number}}$ is never negative

$\sqrt{a^2+b^2}$ is not $\sqrt{a^2} + \sqrt{b^2}$

(for example take $a=3$, $b=4$)

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ provided $a, b$ are both positive

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ provided $a, b$ are both positive ex: simplify

$\sqrt{4x^2}$

$$\sqrt{4x^2} = \sqrt{4}\,\sqrt{x^2} = 2\,|x|$$

$\sqrt{x^2+2x+1}$

$$\sqrt{x^2+2x+1} = \sqrt{(x+1)^2} = |x+1|$$

$\sqrt{x^4+10x^2+25}$

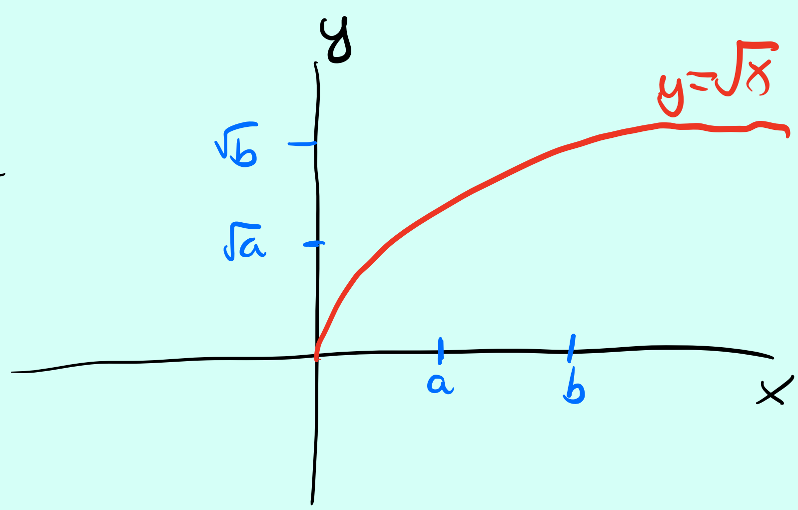
$$\sqrt{x^4+10x+25} = \sqrt{(x^2+5)^2}$$

$= x^2+5$ (here absolute value isn't needed since $x^2+5$ is always non negative)

Square root function:

$y = \sqrt{x}$

| $x$ | $y=\sqrt{x}$ |
|---|---|
| 0 | $\sqrt{0}=0$ |
| 1 | $\sqrt{1}=1$ |
| 4 | $\sqrt{4}=2$ |
| 9 | $\sqrt{9}=3$ |
| 16 | $\sqrt{16}=4$ |



square root preserves inequalities

if $0 \le a \le b$ then $0 \le \sqrt{a} \le \sqrt{b}$

Likewise if $\sqrt{a} \le \sqrt{b}$, then $a \le b$

square roots and equalities

If $\sqrt{a} = b$, then $(\sqrt{a})^2 = b^2$

that is $a = b^2$

solve

$$\sqrt{x+11} = 5$$

notice that $x+11 \geq 0$ otherwise we wouldn't be able to take square roots

we square both sides to obtain

$$\left(\sqrt{x+11}\right)^2 = (5)^2$$

$$x+11 = 25$$

$$x+11-11 = 25-11$$

$\boxed{x = 14}$ notice this value of $x$ satisfies $x+11 \geq 0$

and the original equation

Solve
$$\sqrt{2x^2-8x}=2$$

notice that we need $2x^2-8x\geq 0$ for the square root to make sense. Square both sides

$$\left(\sqrt{2x^2-8x}\right)^2=2^2$$

$$2x^2-8x=4$$

$$2x^2-8x=4$$

$$\frac{1}{2}(2x^2-8x)=\frac{1}{2}\cdot 4$$

$$x^2-4x=2$$

complete squares with $b=-4$, $\frac{b}{2}=-2$ $\left(\frac{b}{2}\right)^2=4$

$$(x-2)^2-4=2$$

$$(x-2)^2=6$$

$x-2=\sqrt{6}$ or $x-2=-\sqrt{6}$

$x=2+\sqrt{6}$ $\quad$ $x=2-\sqrt{6}$

notice that for this value of $x$

$$\sqrt{2x^2-8x}$$
$$=\sqrt{2x(x-4)}$$
$$=\sqrt{2(2+\sqrt{6})(2+\sqrt{6}-4)}$$
$$=\sqrt{2(\sqrt{6}+2)(\sqrt{6}-2)}$$
$$=\sqrt{2(6-4)}$$
$$=\sqrt{4}$$
$$=2$$

$$\sqrt{2x^2-8x}$$
$$=\sqrt{2x(x-4)}$$
$$=\sqrt{2(2-\sqrt{6})(2-\sqrt{6}-4)}$$
$$=\sqrt{2(2-\sqrt{6})(-2-\sqrt{6})}$$
$$=\sqrt{-2(2-\sqrt{6})(2+\sqrt{6})}$$
$$=\sqrt{-2(4-6)}$$
$$=\sqrt{-2(-2)}=\sqrt{4}=2$$

Therefore, when solving an equation involving square roots, one must always check that the solution works

Solve

$\sqrt{x-5} + \sqrt{x} = 5$

we need $x-5 \geq 0$ and $x \geq 0$ for the inequalities to make sense

$$\sqrt{x-5} + \sqrt{x} = 5$$

$$\sqrt{x-5} + \sqrt{x} - \sqrt{x} = 5 - \sqrt{x}$$

$$\sqrt{x-5} = 5 - \sqrt{x}$$

$$(\sqrt{x-5})^2 = (5-\sqrt{x})^2$$

$$x-5 = 25 - 10\sqrt{x} + (\sqrt{x})^2$$

$$x-5 = 25 - 10\sqrt{x} + x$$

$$x-5-x = 25 - 10\sqrt{x} - x$$

$$-5 = 25 - 10\sqrt{x}$$

$$-5+5 = 25 - 10\sqrt{x} + 5$$

$$0 = 30 - 10\sqrt{x}$$

$$0 + 10\sqrt{x} = 30 - 10\sqrt{x} + 10\sqrt{x}$$

$$10\sqrt{x} = 30$$

$$\sqrt{x} = 3$$

$$(\sqrt{x})^2 = 3^2$$

$$x = 9$$

notice that "x" satisfies the original equation since $\sqrt{9-5} + \sqrt{9} = \sqrt{4} + 3 = 2+3 = 5$

solve

$$\sqrt{x+4} + 2 - x = 0$$

we solve for $\sqrt{x-4}$ first

$$\sqrt{x+4} = x - 2$$

square both sides of the equation

$$(\sqrt{x+4})^2 = (x-2)^2$$

$$x+4 = x^2 - 4x + 4$$

$$x = x^2 - 4x$$

$$0 = x^2 - 5x$$

$$0 = x(x-5)$$

so $x = 0$ or $x = 5$

now we check the original equations

$$\sqrt{0+4} + 2 - 0$$

$$= \sqrt{4} + 2$$

$$= 2 + 2$$

$$= 4$$

which is not 0

so $x=0$ does not work

$$\sqrt{5+4} + 2 - 5$$

$$= \sqrt{9} + 2 - 5$$

$$= 3 + 2 - 5$$

$$= 5 - 5$$

$$= 0$$

so $x=5$ does work

# Project 9
## Rational Expressions

A rational expression is a ratio (fraction) of two functions of $x$: for example

$$\frac{x-1}{x^2+1}, \quad \frac{5}{x+3}, \quad \frac{x^2+x+3}{x^2-1}$$

In general we write a rational expression as

$$\frac{f(x)}{g(x)}$$

Many times it is possible to simplify a rational expression using the rules for fractions

$$\frac{ab}{ac} = \frac{b}{c}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

example: simplify

$$\frac{x^2-4}{x-2}$$

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

$$\frac{5x^2-10x}{x^2-4x+4}$$

$$\frac{5x^2-10x}{x^2-4x+4} = \frac{5x(x-2)}{(x-2)^2} = \frac{5x(x-2)}{(x-2)(x-2)} = \frac{5x}{x-2}$$

$$\frac{3}{x+2} + \frac{5}{x-4}$$

$$\frac{3}{x+2} + \frac{5}{x-4}$$

$$= \frac{3}{x+2} \cdot \frac{(x-4)}{(x-4)} + \frac{5}{x-4} \cdot \frac{(x+2)}{(x+2)}$$

$$= \frac{3x-12}{(x+2)(x-4)} + \frac{5x+10}{(x-4)(x+2)}$$

$$= \frac{8x-2}{(x+2)(x-4)}$$

$$\frac{x}{x^2+5x+6} + \frac{x}{x^2-9}$$

$$\frac{x}{x^2+5x+6} + \frac{x}{x^2-9}$$

$$= \frac{x}{(x+2)(x+3)} + \frac{x}{(x-3)(x+3)}$$

$$= \frac{x}{(x+2)(x+3)} \cdot \frac{(x-3)}{(x-3)} + \frac{x}{(x-3)(x+3)} \cdot \frac{(x+2)}{(x+2)}$$

$$= \frac{x^2-3x}{(x+2)(x+3)(x-3)} + \frac{x^2+2x}{(x-3)(x+3)(x+2)}$$

$$= \frac{2x^2-x}{(x+2)(x+3)(x-3)}$$

$$= \frac{x(2x-1)}{(x+2)(x+3)(x-3)}$$

Graphs of rational functions

why is division by 0 not possible?

Suppose we could divide by 0. Then $\frac{1}{0}$ would make sense. Call this number a. So $a = \frac{1}{0}$

Then $0 \cdot a = 0 \cdot \frac{1}{0}$

$$0 = 1$$

which makes no sense!

another explanation....

what happens when you divide by small numbers

$\frac{1}{0.1} = 10$

$\frac{1}{0.01} = 100$

$\frac{1}{0.001} = 1000$

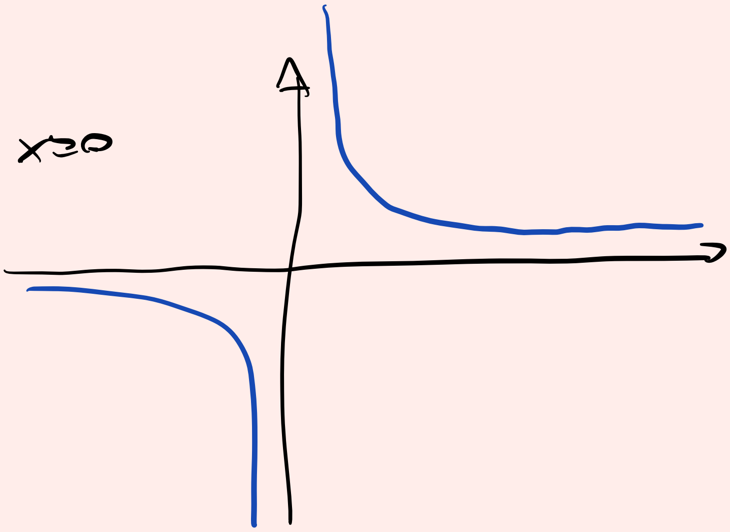
so the smaller the denominator, the larger the ratio becomes.

so in a sense

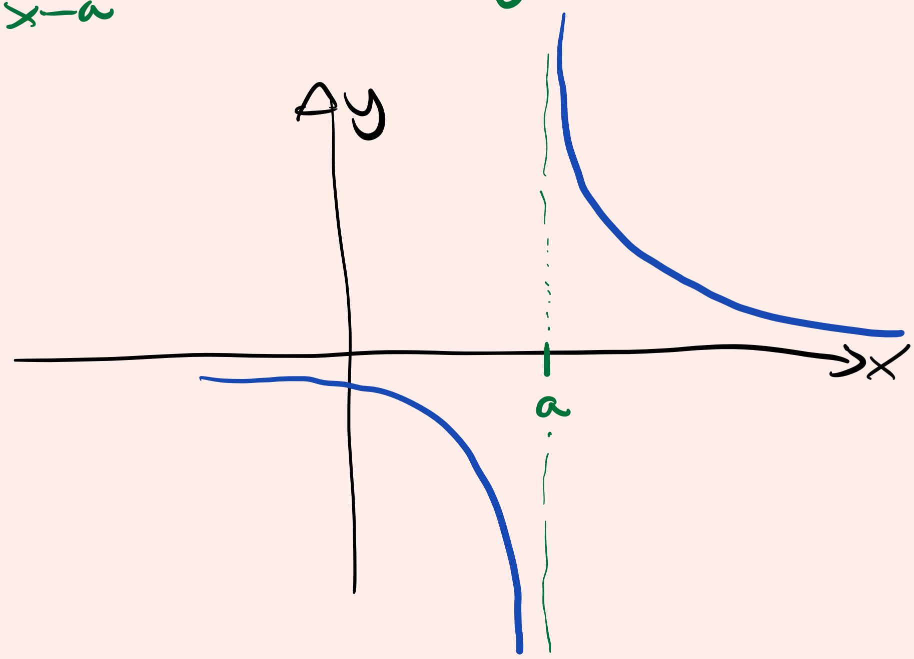
$\frac{1}{0}$ should be infinite!

$y = \frac{1}{x}$

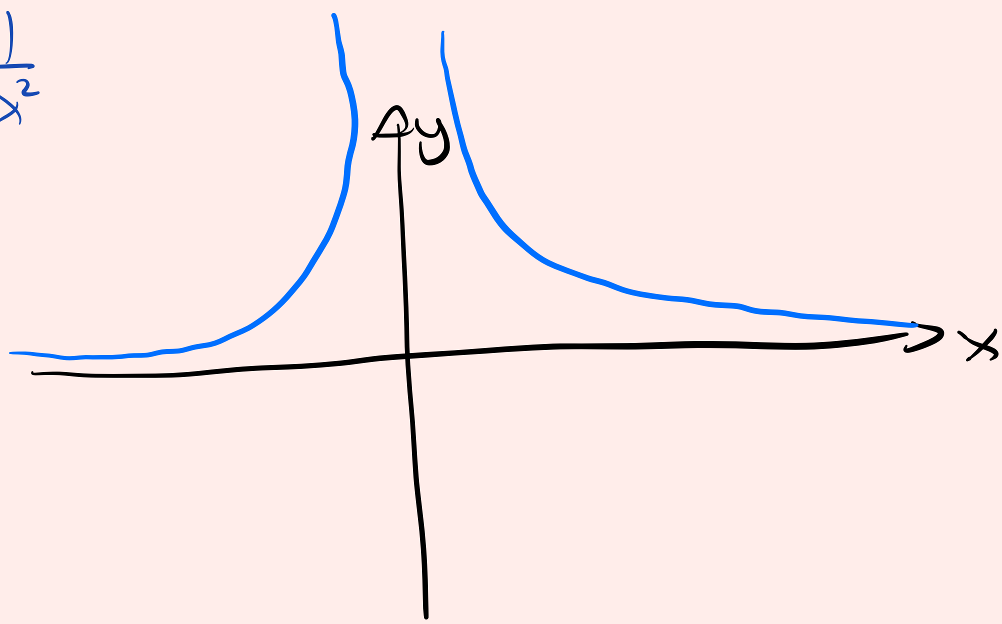
undefined at $x = 0$



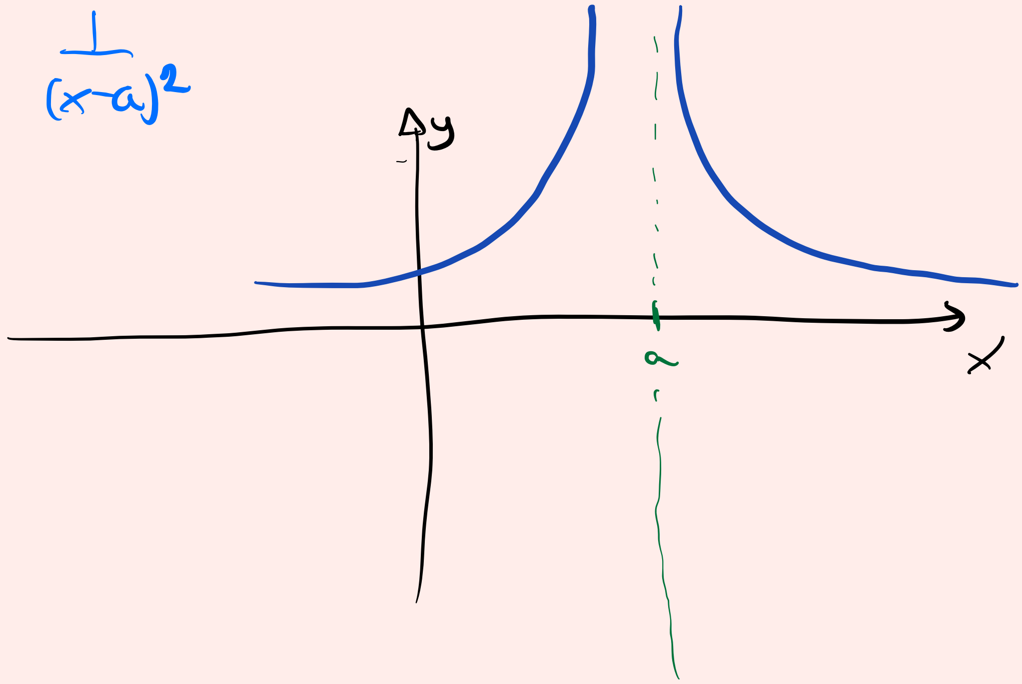
$y = \frac{1}{x-a}$ translated by $a$ units




$y = \frac{1}{x^2}$



$$y = \frac{1}{(x-a)^2}$$




## Division of polynomials

Recall how to divide two whole numbers

$$\frac{21}{4} = \frac{20+1}{4} = \frac{20}{4} + \frac{1}{4} = 5 + \frac{1}{4}$$

multiplying by the denominator both sides of the equation we find

$$21 = 5 \cdot 4 + 1$$

numerator = number · denominator + remainder

For two polynomials $p(x)$, $q(x)$, where the degree of $p(x)$ is higher than that of $q(x)$, we have

$$p(x) = d(x)\, q(x) + r(x)$$

where $d(x)$ is a polynomial with degree equal to degree $p(x)$ − degree $q(x)$ and the degree of $r(x)$ is less than $q(x)$

example:

divide

$$\frac{x^3+x+1}{x-1}$$

we write

$$x^3+x+1 = (x-1)\,d(x) + r(x)$$

since

$\text{degree}(x^3+x+1) = 3$

$\text{degree}(x-1) = 1$

$d(x)$ has degree 2 so we write

$$d(x) = ax^2+bx+c$$

and $r(x)$ has degree less than the degree of $x-1$, so it has degree 0, in other words

$d(x) = d$ is a constant

so

$$x^3+x+1 = (x-1)(ax^2+bx+c) + d$$

$$x^3+x+1 = ax^3+bx^2+cx-ax^2-bx-c+d$$

$$x^3+x+1 = ax^3+(b-a)x^2+(c-b)x+(d-c)$$

Now we equate coefficients

$$\begin{cases} 1 = a \\ 0 = b-a \\ 1 = c-b \\ 1 = d-c \end{cases}$$

The first equation says $\boxed{a=1}$

The second equation says

$$\boxed{b=a=1}$$

The third equation says

$$\boxed{c=1+b=1+1=2}$$

The last equation says

$$\boxed{d=1+c=1+2=3}$$

Hence

$$x^3+x+1=(x-1)(x^2+x+2)+3$$

and therefore

$$\frac{x^3+x+1}{x-1}=\frac{(x-1)(x^2+x+2)+3}{x-1}$$

$$=\frac{(x-1)(x^2+x+2)}{x-1}+\frac{3}{x-1}$$

$$=x^2+x+2+\frac{3}{x-1}$$

Project 10

Rational equations and rationalizing

To solve a rational equation we rewrite it so that there are no denominators in the equation.

solve

$$\frac{1}{4} + \frac{9}{x} = 1$$

notice that "x" can't be 0. we combine terms

$$\frac{1}{4} \cdot \frac{x}{x} + \frac{9}{x} \cdot \frac{4}{4} = 1$$

$$\frac{x}{4x} + \frac{36}{4x} = 1$$

$$\frac{x+36}{4x} = 1$$

$$\frac{x+36}{4x} \cdot 4x = 1 \cdot 4x$$

$$x+36 = 4x$$

$$36 = 3x$$

$$\boxed{12 = x}$$ works since it is not 0.

Solve

$$\frac{3x}{x-4} + \frac{2x}{x+5} + \frac{18x}{x^2+x-20} = 0$$

we need $x-4 \neq 0$, $x+5 \neq 0$ and $x^2+x-20 \neq 0$

$$\frac{3x}{x-4} + \frac{2x}{x+5} + \frac{18x}{(x+5)(x-4)} = 0$$

$$\frac{3x}{x-4} \cdot \frac{x+5}{x+5} + \frac{2x}{x+5} \cdot \frac{x-4}{x-4} + \frac{18x}{(x+5)(x-4)} = 0$$

$$\frac{3x^2+15x}{(x-4)(x+5)} + \frac{2x^2-8x}{(x+5)(x-4)} + \frac{18x}{(x+5)(x-4)} = 0$$

$$\frac{3x^2+15x+2x^2-8x+18x}{(x-4)(x+5)} = 0$$

$$\frac{5x^2+25x}{(x-4)(x+5)} \cdot (x-4)(x+5) = 0 \cdot (x-4)(x+5)$$

$$5x^2+25x = 0$$

$$5x(x+5) = 0$$

so $5x = 0$ or $x+5 = 0$

$x = 0$ or $x = -5$

works

doesn't work for original equation

Solve

$$4 - \frac{3x}{x-9} = \frac{5x-72}{x-9}$$

notice that $x$ can't be 9

$$4 \cdot \frac{x-9}{x-9} - \frac{3x}{x-9} = \frac{5x-72}{x-9}$$

$$\frac{4x-36}{x-9} - \frac{3x}{x-9} = \frac{5x-72}{x-9}$$

$$\frac{4x-36-3x}{x-9} = \frac{5x-72}{x-9}$$

$$\frac{x-36}{x-9} \cdot (x-9) = \frac{5x-72}{x-9} \cdot (x-9)$$

$$x-36 = 5x-72$$

$$x-36+72 = 5x-72+72$$

$$x+36 = 5x$$

$$x+36-x = 5x-x$$

$$36 = 4x$$

$$\frac{36}{4} = x$$

$$9 = x$$

this value doesn't solve original equation

# Rationalizing

Rationalizing an expression involves removing square roots from either the numerator or denominator

example
rationalize $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

rationalize

$\frac{2+x}{\sqrt{x}}$

$$\frac{2+x}{\sqrt{x}} = \frac{2+x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{(2+x)\sqrt{x}}{x}$$

when the denominator involves a difference or sums of square roots, we multiply by the conjugate and use the difference of squares formula

| expression | conjugate |
|---|---|
| $\sqrt{a}-\sqrt{b}$ | $\sqrt{a}+\sqrt{b}$ |
| $\sqrt{a}+\sqrt{b}$ | $\sqrt{a}-\sqrt{b}$ |
| $a-\sqrt{b}$ | $a+\sqrt{b}$ |
| $\sqrt{a}+b$ | $\sqrt{a}-b$ |

rationalize

$$\frac{x}{\sqrt{x}+3}$$

$$\frac{x}{\sqrt{x}+3} = \frac{x}{\sqrt{x}+3} \cdot \frac{\sqrt{x}-3}{\sqrt{x}-3}$$

$$= \frac{x(\sqrt{x}-3)}{(\sqrt{x})^2 - 3^2}$$

$$= \frac{x(\sqrt{x}-3)}{x-9}$$

rationalize

$$\frac{2}{\sqrt{x}-\sqrt{x-2}}$$

$$\frac{2}{\sqrt{x}-\sqrt{x-2}} = \frac{2}{\sqrt{x}-\sqrt{x-2}} \cdot \frac{\sqrt{x}+\sqrt{x-2}}{\sqrt{x}+\sqrt{x-2}}$$

$$= \frac{2(\sqrt{x}+\sqrt{x-2})}{(\sqrt{x})^2 - (\sqrt{x-2})^2}$$

$$= \frac{2(\sqrt{x}+\sqrt{x-2})}{x-(x-2)}$$

$$= \frac{2(\sqrt{x}+\sqrt{x-2})}{x-x+2}$$

$$= \frac{2(\sqrt{x}+\sqrt{x-2})}{2}$$

$$= \sqrt{x}+\sqrt{x-2}$$